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The Finite Irreducible Monomial Linear Groups of Degree 4

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This paper contains an irredundant listing of the finite irreducible monomial subgroups of $GL(4, \mathbb{C})$. The groups are listed up to conjugacy and are given explicitly by generating sets of monomial matrices.

1. INTRODUCTION

Let $M(4)$ denote the group of all monomial matrices in $GL(4) = GL(4, \mathbb{C})$. In [5], the finite irreducible 2-subgroups of $M(4)$ are listed up to $GL(4)$-conjugacy. Here we extend the classification to all of $M(4)$. That is, we provide a list of finite irreducible subgroups of $M(4)$, where any group of this kind is conjugate by an element of $GL(4)$ to one and only one listed group.

Our list is quite explicit, like the one in [5]. An archetype is the list of the finite irreducible p-subgroups of $GL(p)$, p prime, provided by Conlon [2]. Chosen representatives for the conjugacy classes of the groups are arranged into finitely many families, and each representative is specified by a generating set of monomial matrices. As with any list of linear groups of a certain class, it is desirable to have a method by which a given suitably described linear group of the same class may be recognised in the list. However, this issue is beyond the scope of the current paper.

L. G. Kovács has proposed listing all finite irreducible subgroups of $GL(n)$ for $n \leq 4$. This project is motivated by a potential application to soluble quotient algorithms; see [5, p. 1] for a brief discussion. We note a specific problem arising in this context. Short in [10] lists (essentially up to conjugacy in the relevant full symmetric groups) the soluble primitive permutation groups of degree less than 256. One may cast the study of primitive permutation groups with soluble socle, each group necessarily

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having prime-power degree, as the study of irreducible subgroups of $GL(n,p)$, for all primes p and $n \geq 1$. Extension of Short's work to degrees 2^8, 2^9, 3^6, 5^4, and 7^3 would match the work of Dixon and Mortimer [3], who determine the primitive permutation groups of degree less than 1000 with insoluble socle. It is worthwhile, then, to investigate how the results of this paper may be used to list the irreducible monomial (hence soluble) subgroups of $GL(4, p)$ for odd primes p, particularly $p = 5$.

An early study of finite subgroups of $GL(n)$ for $n \leq 4$ is contained in Chapters III, V, and VII of Blichfeldt's book [1]. Blichfeldt considers only subgroups of $SL(n)$, focusing on those that are primitive (a fixed expression in n bounds above the order such a linear group can have). His classification criteria include abstract as well as linear isomorphism. In addition, Blichfeldt gives a primitive linear group whose associated collineation group (central quotient) is simple as a collineation group. Now generally there would seem to be substantial distance to cover in going from a description of subgroups of $SL(n)$ or $PSL(n)$ to a classification by conjugacy of subgroups of $GL(n)$. Moreover, we emphasise that Blichfeldt's treatment of imprimitive groups for $n = 3$ and especially for $n = 4$ is inadequate for our purposes: see, for example, the second paragraph of [1, p. 139].

Another part of the Kovács project is the subject of [6], in which B. Höfling supplies a list of the finite imprimitive nonmonomial subgroups of $GL(4)$ (and also a list of the finite primitive subgroups of $GL(2)$). Subsequent steps in the project are to finish the degree 4 classification, and do a full classification in degrees 2 and 3. Given the achievements of [6] and the present paper, the former step entails listing primitive linear groups. A reference for this is [1, Chap. VII]. The latter step should be easier, since the degrees are prime; for instance, it divides into primitive and monomial cases only. Relevant references here are [1, 2, 6].

We now set down some basics of the main problem to be considered in this paper. In $GL(4)$, denote by $D(4)$ the group of all diagonal matrices, and by S_4 the group of all permutation matrices. We have $M(4) = D(4) \rtimes S_4$. Let π denote projection from $M(4)$ into S_4. A subgroup G of $M(4)$ is an extension of its *diagonal subgroup* $D(4) \cap G$ by its *projection group* $T = \pi G$. If G is irreducible then T is transitive, hence conjugate to one of

$$\begin{aligned}
V_4 &= \langle a, b \mid a^2 = b^2 = 1,\ a^b = a \rangle,\\
C &= \langle c \mid c^4 = 1 \rangle,\\
D &= \langle a, c \mid a^2 = c^4 = 1,\ c^a = c^{-1} \rangle\\
A_4 &= \langle a, b, d \mid a^2 = b^2 = d^3 = 1,\ a^b = a,\ a^d = ab,\ b^d = a \rangle,\\
S_4 &= \langle a, b, d, e \mid a^2 = b^2 = d^3 = e^2 = 1,\ a^b = a,\ a^d = ab,\\
&\qquad b^d = a,\ a^e = a,\ b^e = ab,\ d^e = d^{-1} \rangle,
\end{aligned}$$

where $a = (1,2)(3,4)$, $b = (1,3)(2,4)$, $d = (1,2,3)$, $e = (1,2)$, and $c = ad^2e = (1,2,3,4)$. (Incidentally, this is as far as Blichfeldt gets in his account of the finite irreducible subgroups of $M(4)$, albeit missing that T could be C: see [1, p. 164].) The Sylow 2-subgroups of S_4 are $V_4\langle e\rangle$, $V_4\langle e\rangle^d$, and $V_4\langle e\rangle^{d^2} = D$. Henceforth, by default T stands for any one of the transitive subgroups of S_4 fixed above.

An outline of the rest of the paper follows. In Section 2 we determine the finite subgroups of $D(4)$ normalised by T, for each T. Our construction of extensions in $M(4)$ of such T-modules depends on material presented in Section 3. In Section 4 we give conditions relating to irreducibility of subgroups of $M(4)$. The final list is the union of five sublists, each one corresponding to a particular projection group. Common features of the sublists shape our approach to the conjugacy problem; Section 5 deals with these features. The sublists themselves are compiled in Sections 6–8. Section 9 consolidates previous sections, yielding the final list.

This paper is very much a generalisation of [5]. We continue to use conventions and notation from that earlier work. Also, we sometimes omit or abbreviate a proof similar to a proof in [5], giving the appropriate reference instead.

Finally, we point out that most of the exposition would remain valid if $\mathbb{C}$ were replaced by any field of zero characteristic with all roots of unity.

2. THE FINITE T-SUBMODULES OF $D(4)$

Denote by X the scalars of $GL(4)$, and by Y the subgroup of $D(4)$ whose elements are each fixed by a and inverted by b (where a and b act by conjugation). Further, set $U = Y^{de}$ and $V = Y^d$. Then $D(4) = XYUV$. Let ϖ be a set of primes. We denote $\mathrm{O}_\varpi(D(4))$ by B_ϖ. If $M < D(4)$ then we write $M \cap B_\varpi$ as M_ϖ. A finite subgroup of $D(4)$ is in the torsion subgroup $B = \prod_p B_p$ of $D(4)$, p running over all primes. Note that $B_p = X_p Y_p U_p V_p$, and this is a direct decomposition if and only if $p \neq 2$.

To determine the finite T-submodules of B, and thereby determine all possibilities for diagonal subgroup of a finite supplement to $D(4)$ in $D(4)T$, it is sufficient to list the finite T-submodules of B_p for each prime p. That is the objective of this section.

Submodules of B are given by group generating sets, as in [5]. The generators making up these sets are now defined. For an integer $k \geq -1$, denote by $\omega_{k,p}$ the primitive p^{k+1}th root of unity $\exp(2\pi\sqrt{-1}/p^{k+1})$, and set

$$x_{k,p} = (\omega_{k,p}, \omega_{k,p}, \omega_{k,p}, \omega_{k,p}),$$

$$y_{k,p} = \left(\omega_{k,p}, \omega_{k,p}, \omega_{k,p}^{-1}, \omega_{k,p}^{-1}\right),$$
$$u_{k,p} = y_{k,p}^{de}, \qquad v_{k,p} = y_{k,p}^{d}.$$

Note that X_p, Y_p, U_p, and V_p are all isomorphic to C_p^∞; that is, $X_p = \varinjlim \langle x_{k,p} \rangle$, $Y_p = \varinjlim \langle y_{k,p} \rangle$, $U_p = \varinjlim \langle u_{k,p} \rangle$, and $V_p = \varinjlim \langle v_{k,p} \rangle$, the direct limits taken with respect to k. The action of S_4 on group generators of B is set out below:

$$\begin{array}{lll}
y_{k,p}^a = y_{k,p} & u_{k,p}^a = u_{k,p}^{-1} & v_{k,p}^a = v_{k,p}^{-1}, \\
y_{k,p}^b = y_{k,p}^{-1} & u_{k,p}^b = u_{k,p} & v_{k,p}^b = v_{k,p}^{-1}, \\
y_{k,p}^c = v_{k,p} & u_{k,p}^c = u_{k,p}^{-1} & v_{k,p}^c = y_{k,p}^{-1}, \\
y_{k,p}^d = v_{k,p} & u_{k,p}^d = y_{k,p} & v_{k,p}^d = u_{k,p}, \\
y_{k,p}^e = y_{k,p} & u_{k,p}^e = v_{k,p} & v_{k,p}^e = u_{k,p}.
\end{array}$$

To make the translation from the notation introduced above to the notation of [5], drop the second subscript p when $p = 2$.

We collect here some miscellaneous module-theoretic facts. It is a consequence of [5, Proposition 1.1.2] that two finite T-submodules of B_p are isomorphic if and only if they are equal, for any prime p and T of order 4. Hence the same statement holds for all T and with B_p replaced by B. Recall also from [5] that a finite nontrivial T-submodule of B_2 contains $\Omega_1(X_2) = \Omega_1(Y_2) = \Omega_1(U_2) = \Omega_1(V_2) = \langle x_{0,2} \rangle = \langle x_{1,2} y_{1,2} u_{1,2} v_{1,2} \rangle$.

Until further notice, M stands for an arbitrary finite T-submodule of B. We say that M is cyclic if it is a cyclic group.

LEMMA 2.1. (i) *If $T = V_4$ or $T = C$ then M is a faithful T-module, or is scalar, or $|\mathsf{C}_T(M)| = 2$.*

(ii) *If $T = D$ then M is faithful or $\mathsf{C}_T(M) \geq \langle ac, b \rangle$, with the first possibility excluded when M is cyclic.*

(iii) *If $T = A_4$ or $T = S_4$ then M is faithful or scalar, with the first possibility excluded when M is cyclic.*

Proof. The kernel of the action of T on M is a normal subgroup of T, with abelian quotient when M is cyclic. All claims follow readily from these observations. ∎

2.1. *The finite cyclic T-submodules of B_2*

In this subsection M is cyclic and in B_2. Define $M(i, j, k, l)$ to be the cyclic subgroup of B_2 generated by $x_{i,2} y_{j,2} u_{k,2} v_{l,2}$. Suppose M is not a faithful T-module. By Lemma 2.1, if $T = A_4$ or $T = S_4$ then $M \leq X_2$. If

$T \le D$ then, again by Lemma 2.1, M is S_4-conjugate to a T-submodule of X_2U_2. At the end of [5, Section 1.2] there is a discussion about the finite normal subgroups of $C_{2^\infty} \operatorname{wr} C_2$ lying in the base group of this wreath product, from which one may deduce the T-submodule structure of X_2U_2. The cyclic submodules in particular are

$$M(i, -1, -1, -1), \qquad i \ge -1;$$
$$M(-1, -1, i, -1), \quad M(i+1, -1, 1, -1), \quad M(1, -1, i+1, -1), \qquad i \ge 1. \quad (1)$$

Suppose M is a faithful V_4-module, so that $M \cong C_{2^{i+2}}$ for some $i \ge 1$. There are six inequivalent faithful representations of V_4 in $\operatorname{Aut}(M)$, and therefore six distinct possibilities for M at each order 2^{i+2}. Explicitly, these are

$$\begin{array}{ll} M(-1, 1, i+1, -1), & M(-1, -1, i+1, 1), \\ M(-1, i+1, 1, -1), & M(-1, i+1, -1, 1), \\ M(-1, 1, -1, i+1), & M(-1, -1, 1, i+1). \end{array} \quad (2)$$

At each order, these submodules are all S_4-conjugate.

Suppose M is a faithful C-module. As in the previous paragraph, we argue that

$$\begin{array}{ll} M(i+2, 1, 2, -1), & M(i+2, -1, 2, 1), \\ M(2, 1, i+2, -1), & M(2, -1, i+2, 1) \end{array} \quad (3)$$

constitute all possibilities for M at order 2^{i+3}, $i \ge 1$. The first and second of these modules are conjugate by a, as are the third and fourth; however, the first and third are not S_4-conjugate.

2.2. *The finite noncyclic T-submodules of B_2*

We transfer to this paper the submodule labelling scheme employed in Section 3.1 and Chapter 4 of [5] (with the necessary adjustment to the notation for generators).

For $T = C$ or $T = D$, the finite noncyclic T-submodules of B_2 are given in [5, Theorem 4.4; Lemma 4.5]. For $T = V_4$, the submodules containing $\Omega_1(X_2Y_2) = \langle x_{1,2}y_{1,2}, u_{1,2}v_{1,2} \rangle$ are specified in [5, Theorem 3.1.8]; all others are obtained as $\langle d \rangle$-conjugates of these. To list the finite noncyclic A_4-submodules M of B_2, we turn to the proof of [5, Proposition 3.1.12]. Displayed there is the orbit of each finite V_4-submodule of B_2 containing $\Omega_1(X_2Y_2)$ under the action of S_4. Of course, the S_4-orbit of M has length at most 2. Inspection of the orbit display then shows that in fact M must be normalised by S_4. Therefore, we have the following proposition.

PROPOSITION 2.2.1. *For $T = A_4$ and $T = S_4$,*

$$\left.\begin{array}{l} F(i,j,j,j,0,0) \\ F(i,j,j,j,0,0,-1,-1) \\ F(i,j,j,j,0,0,0,-1) \\ F(i,j,j,j,0,1,0) \end{array}\right\} \qquad i,j \geq 1,$$

$$\left.\begin{array}{l} F(i,j,j,j,1,1,1) \\ F(i,j,j,j,1,1,2,1) \\ F(i,j,j,j,1,1,2,-1) \end{array}\right\} \qquad i = j = 0 \quad \textit{or} \quad i,j \geq 1$$

is a complete list of all finite noncyclic T-submodules of B_2.

2.3. *The finite T-submodules of B_p, $p \neq 2$*

In this subsection, p is an odd prime. We use repeatedly [4, Theorem 3.1] without reference.

Since b acts trivially on subquotients of X_p and invertingly on subquotients of Y_p, we see that the finite V_4-submodules of X_pY_p are just the "Cartesian" submodules $\langle x_{i,p}, y_{j,p}\rangle$, for i,j ranging over the integers subject to $i,j \geq -1$. Similarly, the finite V_4-submodules of U_pV_p are the $\langle u_{k,p}, v_{l,p}\rangle$, $k,l \geq -1$. After considering how a acts on subquotients of X_pY_p and U_pV_p, we conclude that a finite V_4-submodule of B_p is one of

$$F(i,j,k,l;p) = \langle x_{i,p}, y_{j,p}, u_{k,p}, v_{l,p}\rangle, \qquad i,j,k,l \geq -1.$$

Note the allowance of negative order parameters i,j,k,l, in contrast to the allowance of non-negative order parameters only for the noncyclic V_4-submodules $F(i,j,k,l,\delta,\xi,\alpha)$ of B_2. Distinct labels, as ever, correspond to distinct submodules.

To solve the conjugacy problem in Section 6, we need to know the orbit of each finite T-submodule of $B_{2'}$ under action by various subgroups of $\mathsf{N}_{S_4}(T)$. We first set up some terminology to facilitate the description of these orbits in the most complicated case, $T = V_4$.

$F(i,j,k,l;p)$ is called *homogeneous*, *partially homogeneous*, or *inhomogeneous*, according as $j = k = l$, precisely two of j,k,l are equal, or j,k,l are all distinct, respectively.

PROPOSITION 2.3.1. *The finite A_4-submodules of B_p, p odd, are precisely the homogeneous V_4-submodules of B_p. These are also the finite S_4-submodules of B_p.*

Proof. Let M be a finite V_4-submodule of B_p such that $M^d = M$. Since $Y_p^d = V_p$ and $Y_p^{d^2} = U_p$, it follows that $M \cap Y_p$, $M \cap U_p$, $M \cap V_p$ are all conjugate and hence have the same order. That is, M is homogeneous. As such, $M^e = M$, and M is also an S_4-submodule of B_p. ∎

Let M be a finite V_4-submodule of $B_{2'}$. We say that M is

type 1, if, for all p, M_p is homogeneous ($\mathsf{N}_{S_4}(M) = S_4$);

type 2, if, for all p, M_p is either homogeneous or partially homogeneous of the form $F(i, j, k, k; p)$, and not all M_p are homogeneous ($\mathsf{N}_{S_4}(M) = V_4\langle e\rangle$);

type 3, if either M has an inhomogeneous submodule $F(i, j, k, l; p)$, $j < k < l$, where p is the least prime such that M_p is inhomogeneous, or M has no inhomogeneous submodules but has partially homogeneous submodules $F(i, j, k, k; q)$ and $F(i', j', j', k'; r)$, where q is the least prime such that M_q is partially homogeneous, and r is the least prime greater than q such that M_r is partially homogeneous with $\mathsf{N}_{S_4}(M_r) \neq V_4\langle e\rangle$ ($\mathsf{N}_{S_4}(M) = V_4$).

We broaden this submodule categorisation by saying that M is type 2^{d^σ} or type 3^{d^σ}, $0 \leq \sigma \leq 2$, if M is the d^σ-conjugate of a submodule of type 2 or 3, respectively. If M is type 1 then M is faithful or scalar; if M is type 2^{d^σ} then M is faithful or $\mathsf{C}_{V_4}(M) = \langle a\rangle^{d^\sigma}$; if M is type 3^{d^σ} then M is faithful.

THEOREM 2.3.2. (i) *M is S_4-conjugate to one and only one element in the list consisting of all types* 1–3 *finite V_4-submodules of $B_{2'}$.*

(ii) *For each σ, $0 \leq \sigma \leq 2$, M is $V_4\langle e\rangle^{d^\sigma}$-conjugate to one and only one element in the list consisting of all types* 1, 2^{d^σ}, $2^{d^{\sigma+1}}$, *and* 3^{d^τ} *finite V_4-submodules of $B_{2'}$*, $0 \leq \tau \leq 2$.

Proof. First note that $\mathsf{N}_{S_4}(M) = \bigcap_p \mathsf{N}_{S_4}(M_p)$ is S_4, or a Sylow 2-subgroup $V_4\langle e\rangle^{d^\sigma}$ of S_4, or V_4 (A_4 does not appear, by Proposition 2.3.1). In the first situation, M is type 1. In the second situation, M is $\langle d\rangle$-conjugate to a submodule of type 2. In the third situation, either M has an inhomogeneous submodule, or has no inhomogeneous submodules but has at least two partially homogeneous submodules with different S_4-normalisers. Then it is not difficult to see that M is S_4-conjugate to one of the (canonically defined) type 3 submodules.

Submodules with S_4-normalisers that are not conjugate cannot themselves be S_4-conjugate. Obviously two different submodules of type 1 are not S_4-conjugate. Conjugation by an element of $S_4 \setminus V_4\langle e\rangle$ sends a submodule of type 2 to a submodule not of type 2. Let M be a type 3 submodule with inhomogeneous submodule M_p, where p is the least prime such that M_p is inhomogeneous, and suppose M^s is type 3 for some

$s \in S_4$. By the minimal choice of p, and since $(M_p)^s = (M^s)_p$, this means that $F(i,j,k,l;p)^s = F(i',j',k',l';p)$ for some i,j,k,l and i',j',k',l' such that $j < k < l$ and $j' < k' < l'$. Hence $s \in V_4$. A similar argument shows that if M is type 3 with no inhomogeneous submodules and M^s is also type 3, then $s \in V_4\langle e\rangle \cap V_4\langle e\rangle^d = V_4$. We have proved (i).

Given (i) and the fact that $S_4 = V_4\langle e\rangle^{d^\sigma}\langle d\rangle$, a finite V_4-submodule of $B_{2'}$ is $V_4\langle e\rangle^{d^\sigma}$-conjugate to a submodule of type 1, type 2^{d^τ}, or type 3^{d^τ}, for some τ, $0 \leq \tau \leq 2$. Also, a type $2^{d^{\sigma+2}}$ submodule is conjugate by $d^\sigma e$ to a type $2^{d^{\sigma+1}}$ submodule. Suppose M and N are both type 2 or type 3 submodules such that M^{d^μ} and N^{d^ν} are $V_4\langle e\rangle$-conjugate. Then $M = N$ by (i) and therefore $d^{\mu-\nu}$ or $d^{\mu+\nu}e$ is in $\mathrm{N}_{S_4}(M)$. That is, $\mu = \nu$, or $\mu = -\nu$ and M is type 2. In the latter event, if $\mu \neq 0$ then one of M^{d^μ} or M^{d^ν} is type 2^{d^2}. This proves (ii) for $\sigma = 0$. For $\sigma = 1$ or $\sigma = 2$, a complete and irredundant list of $V_4\langle e\rangle^{d^\sigma}$-conjugacy class representatives of the finite V_4-submodules of $B_{2'}$ is obtained by taking the d^σ-conjugates of all modules in the previously obtained list of $V_4\langle e\rangle$-conjugacy class representatives. So we have proved all of (ii). ∎

It remains to determine the finite T-submodules of $B_{2'}$ for $T = C$ and $T = D$. We have $B_p = (X_pU_p) \times (Y_pV_p)$ as C-modules, but whereas X_p and U_p are each normalised by C, neither Y_p nor V_p are: $Y_p^c = V_p$ and $V_p^c = Y_p$. Action by b is trivial on X_pU_p and inverting on Y_pV_p, so we see by the usual reasoning that a finite C-submodule of B_p is the direct product of some $\langle x_{i,p}, u_{k,p}\rangle$ and a C-submodule of Y_pV_p. The finite subgroups of Y_pV_p are

$$\left.\begin{array}{l}\langle y_{j+r,p}v_{l+r,p}^s, y_{j,p}, v_{l,p}\rangle \\ \langle y_{j,p}, v_{l,p}\rangle\end{array}\right\} \quad j,l \geq -1,\ r \geq 1,$$

where $1 \leq s \leq p^r - 1$ and $(s,p) = 1$. For one of these subgroups to be normalised by C, it is necessary that $l = j$. Certainly $\langle y_{j,p}, v_{j,p}\rangle$ is a C-submodule of Y_pV_p. Now $\langle y_{j+r,p}v_{j+r,p}^s, y_{j,p}, v_{j,p}\rangle$ is normalised by C if and only if $v_{j+r,p}^{1+s^2} \in \langle v_{j,p}\rangle$; that is, $s^2 \equiv -1 \bmod p^r$. Take the p-adic expansion $a_0 + a_1p + a_2p^2 + \cdots + a_{r-1}p^{r-1}$ of s, where $0 \leq a_i \leq p-1$ and $a_0 \neq 0$. If $s^2 \equiv -1 \bmod p^r$ then $a_0^2 \equiv -1 \bmod p$. If $p \equiv 3 \bmod 4$, this cannot happen (-1 is a quadratic non-residue mod p). Assume, then, that $p \equiv 1 \bmod 4$. There are exactly two choices for a_0. Once a_0 has been fixed, $a_1, a_2, \ldots, a_{r-1}$ are uniquely determined, and hence so too is s. For if a_0 has been chosen and $r \geq 2$, then a_1 satisfies $(a_0 + a_1p)^2 \equiv -1 \bmod p^2$, which implies $a_1 \equiv -(1 + a_0^2)/2a_0p \bmod p$. More generally, suppose $1 \leq m \leq r-1$ and that $a_{m-1}, \ldots, a_1$ have all been found. Then for $\alpha = a_0 + a_1p + \cdots + a_{m-1}p^{m-1}$ we have $(\alpha + a_mp^m)^2 \equiv -1 \bmod p^{m+1}$, so that $a_m \equiv -(1+\alpha^2)/2\alpha p^m \bmod p$. In this way we can always calculate the two square roots of $-1 \bmod p^r$, given $r \geq 1$. Thus we obtain the finite C-submodules

$$C(i, j, k, r, \xi; p) = \langle x_{i,p}, u_{k,p}, y_{j+r,p} v_{j+r,p}^{\xi s}, y_{j,p}, v_{j,p} \rangle, \qquad \xi \in \{-1, 1\},$$

of B_p for $p \equiv 1 \bmod 4$, where s is a fixed square root of -1 in $\mathbb{Z}_{p^r}$.

We summarise our findings below.

THEOREM 2.3.3. *The following is a complete list of the finite C-submodules of B_p, p odd,*

$$C(i, j, k, r, \xi; p) \quad p \equiv 1 \bmod 4 \textit{ only},$$
$$F(i, j, k, j; p) \quad \textit{all } p,$$

where ξ ranges over $\{-1, 1\}$, and i, j, k, r range over the integers subject to $i, j, k \geq -1$ and $r \geq 1$.

Note that $C(i, j, k, r, 1; p)^a = C(i, j, k, r, -1; p)$. Theorem 2.3.3 then leads directly to the next two corollaries.

COROLLARY 2.3.4. *A finite subgroup M of $B_{2'}$ is a D-module if and only if, for each odd prime p, $M_p = F(i_p, j_p, k_p, j_p; p)$ for some $i_p, j_p, k_p \geq -1$.*

COROLLARY 2.3.5. *A finite C-submodule of $B_{2'}$ either is normalised by D, or is conjugate by an element of $\mathsf{N}_{S_4}(C) = D$ to M such that if p is the least prime (congruent to 1 mod 4) for which M_p is not normalised by D, then $M_p = C(i, j, k, r, 1; p)$ for some $i, j, k \geq -1$ and $r \geq 1$. Moreover, in the list of all C-module of these kinds, no two distinct ones are D-conjugate.*

3. THE EXTENSION PROBLEM

A major part of the endeavour in this paper is solving the extension problem: constructing, up to conjugacy, T-extensions in $M(4)$ of finite T-submodules of B. This section establishes some machinery to assist in that task. The divisibility of the abelian group $\mathbb{C}^\times$ is crucial in proofs.

We assume the standard description of cohomology of finite cyclic groups, for which see [8, p. 122, Theorem 7.1]. As a special case of this, note that if $G = \langle g \rangle$ has order n and M is a G-module then we have an isomorphism

$$m + M^{1-g} \in \left(\ker\left(1 + g + \cdots + g^{n-1}\right) \text{ on } M\right)/M^{1-g} \mapsto [\delta] \in H^1(G, M),$$

where the derivation δ is defined by $\delta(g^i) = m^{1+g+\cdots+g^{i-1}}$, $1 \leq i \leq n$. If G is any group then it is very well known that as δ runs over a set of representatives for the elements of $H^1(G, M)$, $\{g\delta(g) \mid g \in G\}$ runs over a set of representatives for the conjugacy classes of complements of M in $M \rtimes G$.

LEMMA 3.1. *Let T be a cyclic subgroup of D. For all $i \geq 1$,*

(i) $H^{2i}(T, D(4)) = 0$.

(ii) *If $T = \langle ac \rangle$ or $T = \langle ac^3 \rangle$ then $H^{2i-1}(T, D(4))$ is isomorphic to a subgroup of $\Omega_1(B_2)$ of order* 4; *otherwise, $H^{2i-1}(T, D(4)) = 0$.*

Proof. Suppose $T = \langle a \rangle$. Then $1 - a$ has kernel XY and image $U^2V^2 = UV$ in $D(4)$; $1 + a$ has kernel UV and image XY in $D(4)$. Thus $H^{2i-1}(T, D(4)) \cong UV/UV \cong 0$ and $H^{2i}(T, D(4)) \cong XY/XY \cong 0$. The proofs for $T = \langle b \rangle$, $T = \langle ab \rangle$, and $T = C$ are similar. We verify that $\ker(1 - ac^i) \leq D(4)^{1+ac^i}$ for $i = 1$ and $i = 3$, and also calculate that

$$H^{2i-1}(\langle ac \rangle, D(4)) \cong \left\langle \left[x_{1,2}^{-1} u_{1,2} \right], \left[x_{2,2}^{-1} y_{2,2} u_{2,2} v_{2,2} \right] \right\rangle,$$

$$H^{2i-1}(\langle ac^3 \rangle, D(4)) \cong \left\langle \left[x_{1,2} u_{1,2} \right], \left[x_{2,2} y_{2,2} u_{2,2} v_{2,2}^{-1} \right] \right\rangle,$$

where $[-]$ denotes cosets modulo $D(4)^{1-ac}$ and modulo $D(4)^{1-ac^3}$, as appropriate. ∎

By the discussion before Lemma 3.1, the following is an immediate consequence of the lemma.

COROLLARY 3.2. *If T is a cyclic subgroup of D then a complement of $D(4)$ in $D(4)T$ is conjugate to a subgroup of B_2T.*

Natural surjection of a group onto a quotient gives rise to the inflation homomorphism in its cohomology (see [8, p. 347]). Inflation often affords a helpful reduction in the calculation of cohomology, by means of the next lemma.

LEMMA 3.3. *Let $n \geq 1$ and suppose G is a group with normal subgroup N, and M is a G-module such that $H^i(N, M) = 0$ for all i, $1 \leq i \leq n$. Then inflation from $H^i(G/N, M^N)$ to $H^i(G, M)$ is an isomorphism for all i, $1 \leq i \leq n$.*

Proof. This is just [8, p. 355, Exercise 3]. ∎

PROPOSITION 3.4. (i) $H^1(V_4, D(4)) = 0$.

(ii) $H^1(\langle ac, b \rangle, D(4)) \cong \Omega_1(X_2U_2)$.

(iii) $H^1(D, D(4)) \cong \Omega_1(X_2)$.

(iv) $H^1(A_4, D(4)) \cong \Omega_1(X_3)$.

(v) $H^1(S_4, D(4)) \cong \Omega_1(X_2)$.

Proof. We use Lemma 3.3 throughout. For $G = V_4$ and $G = \langle ac, b\rangle$, we may take $N = \langle b\rangle$ by Lemma 3.1, so that $H^1(V_4, D(4)) \cong U/U^2 \cong 0$ and $H^1(\langle ac, b\rangle, D(4)) \cong H^1(\langle ac\rangle, XU) \cong \ker(1 + ac)$ on XU. By the same method, $H^1(D, D(4)) \cong \ker(1 + a)$ on X, and $H^1(A_4, D(4)) \cong \ker(1 + d + d^2)$ on X. Finally,

$$H^1(S_4, D(4)) \cong H^1(S_3, X) = \mathrm{Hom}(S_3/S_3', X),$$

and we are done. ∎

Parts of Lemma 3.1 and Proposition 3.4(i) and (iii) are proved in [5, Lemma 2.2].

COROLLARY 3.5. *$[\delta] \in H^1(T, D(4))$ has representative δ such that* $\mathrm{im}\,\delta \le \Omega_1(X_2)$ *if $T = D$ or $T = S_4$,* $\mathrm{im}\,\delta \le \Omega_1(X_2U_2)$ *if $T = \langle ac, b\rangle$, and* $\mathrm{im}\,\delta \le \Omega_1(X_3)$ *if $T = A_4$.*

Furthermore, a complement of $D(4)$ in $D(4)T$ is conjugate to a subgroup of $\Omega_1(X_2)S_4$ if $T = S_4$, (more generally) to a subgroup of B_2T if $T \le D$, and to a subgroup of B_3T if $T = A_4$.

Proof. The claims about $\mathrm{im}\,\delta$ are immediate from Proposition 3.4, after applying the isomorphism described before Lemma 3.1 (note also that inflation does not alter images). The claim on complements is trivial for $T = V_4$ by Proposition 3.4(i); for the other T, it is stated in Corollary 3.2 or is plain by the previous part. ∎

We now present some results for second cohomology.

PROPOSITION 3.6. *If T is A_4, S_4, or a subgroup of D, then $H^2(T, D(4)) = 0$.*

Proof. For $T \ne S_4$, this follows from Lemmas 3.1 and 3.3, and Proposition 3.4. We obtain $H^2(S_4, D(4)) \cong H^2(S_3, X)$ in the same manner. Then by the universal coefficient theorem (see [9, p. 349, 11.4.18]),

$$\begin{aligned} H^2(S_3, X) &\cong \mathrm{Ext}(S_3/S_3', X) \oplus \mathrm{Hom}\big(H_2(S_3), X\big) \\ &\cong \mathrm{Ext}(C_2, X) \oplus \mathrm{Hom}(1, X) \\ &= 0, \end{aligned}$$

which proves the proposition. ∎

COROLLARY 3.7. *For T as in Proposition 3.6 and all primes p, $H^2(T, B_p) = 0$.*

Proof. Since $D(4)/B$ is a divisible torsion-free abelian group, $H^i(T, D(4)/B) = 0$ for all $i \ge 1$ by [8, p. 117, Corollary 5.4]. Applying this fact to the cohomology long exact sequence arising from the short exact sequence

$$1 \to B \overset{\text{inc.}}{\to} D(4) \overset{\text{proj.}}{\to} D(4)/B \to 1$$

of T-modules (cf. the proof of [5, Theorem 2.5]), we get $H^2(T, B) = H^2(T, D(4))$. Since $B = \Pi_p B_p$ and $H^n(T, -)$ is an additive functor, $H^2(T, B_p) = 0$ by Proposition 3.6. ∎

PROPOSITION 3.8. *Let M be a finite T-submodule of B, where T is a transitive subgroup of S_4.*

(i) *If $T = V_4, C, D$, or S_4 then $H^2(T, M) \cong H^2(T, M_2)$.*

$$\text{(ii)}\quad H^2(A_4, M) \cong \begin{cases} H^2(A_4, M_2) \oplus \mathbb{Z}_3 & \text{if } M_3 \cap X \neq 1 \\ H^2(A_4, M_2) & \text{if } M_3 \cap X = 1. \end{cases}$$

Proof. The Schur–Zassenhaus theorem (see [9, p. 253, 9.1.2]) is used several times. By additivity and the theorem, $H^2(T, M) \cong H^2(T, M_2) \oplus H^2(T, M_3)$. Then (i) follows for $T \leq D$. The reasoning that shows $H^2(S_4, D(4)) = 0$ in the proof of Proposition 3.6 carries over to give $H^2(S_4, M_3) = 0$. Finally, (ii) follows from $H^2(A_4, M_3) \cong (M_3 \cap X)/(M_3 \cap X)^3$. ∎

The next theorem strengthens [5, Theorem 2.5]. It is proved along more or less the same lines, employing Lemma 3.1, Propositions 3.4, 3.6, and 3.8, and Corollary 3.5.

THEOREM 3.9. *Let T be a transitive subgroup of S_4 and M a finite T-submodule of B. Denote by n the number of $D(4)T$-conjugacy classes of finite subgroups of $D(4)T$ with diagonal subgroup M and projection group T.*

(i) *If $M_2 \neq 1$ and $T \neq A_4$, then $n = |H^2(T, M_2)|$.*

(ii) *If $T = A_4$ then $n = 3|H^2(T, M_2)|$.*

(iii) *If $M_2 = 1$ then $n = 2$ for $T = D$ or $T = S_4$, and $n = 1$ for $T = V_4$ or $T = C$.*

Theorem 3.9 is not as significant in solving the extension problem for irreducible subgroups of $M(4)$ with projection group A_4 or S_4 as it is in solving that problem for irreducible subgroups of $B_2 D$. An instructive illustration of this concerns the finite irreducible groups in $B_2 V_4$ with diagonal subgroup $M = F(i, j, k, l, 0, 0)$. If we can show that any such group is $D(4)V_4$-conjugate to one of $|H^2(V_4, M)| = 32$ explicitly constructed V_4-extensions of M, then we avoid having to prove in $\binom{32}{2}$ cases that there is no $D(4)V_4$-conjugacy between the extensions: this follows from Theorem 3.9. We have not been able to devise a method for calculating $|H^2(T, M)|$ for all M and $T = A_4$ or $T = S_4$. Fortunately, this number is always small—never more than 4, as we see in Propositions 7.4 and 8.2. A case-by-case verification that extensions are pairwise non-conjugate in $D(4)T$ is consequently not onerous.

The Sylow 2-subgroups of $GL(4)$ are all conjugate to $B_2 D$. We show next that a Sylow 2-subgroup of $M(4)$ is in fact conjugate by a monomial matrix to $B_2 D$. Recall from [5] the notation $G \sim H$, which denotes that $G, H \leq GL(4)$ are conjugate. If $G^r = H$ for some $r \in R \leq GL(4)$, then we may write $G \sim_r H$ or $G \sim_R H$.

PROPOSITION 3.10. *If S is a maximal 2-subgroup of $M(4)$ then $S \sim_{M(4)} B_2 D$. Therefore, every 2-subgroup of $M(4)$ is $M(4)$-conjugate to a subgroup of $B_2 D$.*

Proof. We assume $S \leq D(4)D$, since an S_4-conjugate of S is in $D(4)D$. Also, by maximality, $S \cap D(4) = B_2$. By Corollary 3.7, S splits over B_2. A complement of B_2 in S, as a complement of $D(4)$ in $D(4)\pi S$, is $D(4)D$-conjugate to a subgroup of $B_2 D$, by Corollary 3.5. Thus $S \sim_{M(4)} B_2 D$. ∎

The last result of this section is elementary yet vital, and is basic in our solution of the extension problem.

THEOREM 3.11. *Let G be a finite subgroup of $M(4)$, with diagonal subgroup M and projection group T. Then, up to conjugacy in $M(4)$,*

(i) *$G = HK \ltimes M_{\{2,3\}'}$ if $T = A_4$, where H is an extension of M_2 by V_4 in $B_2 V_4$ and K is an extension of M_3 by $\langle d \rangle$ in $D(4)\langle d \rangle$;*

(ii) *$G = H \ltimes M_{2'}$ if $T = S_4$, where H is an extension of a subgroup K by $\langle e \rangle$ in $M(4)$, with $\pi K = A_4$ and $K \cap B = M_2$;*

(iii) *$G = H \ltimes M_{2'}$ if $T = V_4$, C, or D, where H is an extension of M_2 by T in $B_2 T$.*

Proof. Suppose $T = A_4$. Since $M/M_2 M_3$ is a normal Hall subgroup of $G/M_2 M_3$, by Schur−Zassenhaus there exists $\hat{G} \leq G$ such that $\hat{G} \cap M = M_2 M_3$ and $\hat{G}M = G$. Thus $\hat{G} \cap M_{\{2,3\}'} = 1$ and $\hat{G}M_{\{2,3\}'} = G$. By Proposition 3.10, we may assume $\hat{G}$ has a Sylow 2-subgroup H lying in $B_2 V_4$, with diagonal subgroup M_2 (naturally) and projection group V_4. Let K be a Sylow 3-subgroup of $\hat{G}$ (and thus of G) with $\pi K = \langle d \rangle$. Now $K \cap B = M_3$. Also $HM_3 \trianglelefteq \hat{G}$, since $HM_3/M_2 M_3$ is the Sylow 2-subgroup of $\hat{G}/M_2 M_3$. From this it may be seen that H and K permute. Thus $\hat{G} = HK$ and (i) is proved.

By Schur−Zassenhaus and Lemma 3.3 we have $H^2(G/M_{2'}, M_{2'}) \cong H^2(G/M, M_{2'})$. By Proposition 3.8(i), $H^2(S_4, M_{2'}) = 0$. Hence G splits over $M_{2'}$ if $T = S_4$. The rest of (ii) is trivial.

If $T \le D$ then as in the previous case we see that G splits over $M_{2'}$. Thus (iii) follows from Proposition 3.10. ∎

4. IRREDUCIBILITY

In Theorem 4.2 below, we give necessary and sufficient conditions for irreducibility of finite subgroups of $M(4)$. This theorem depends on the next result, which removes superfluous restrictions in [5, Proposition 1.3.6].

PROPOSITION 4.1. *Let G be a finite irreducible subgroup of $GL(4)$ with an abelian normal subgroup A of index 4. Then there exists $m \in GL(4)$ such that $A^m = B \cap G^m$ and $G^m \le M(4)$.*

THEOREM 4.2. *Let G be a finite subgroup of $M(4)$ with diagonal subgroup M and transitive projection group T. If M is a faithful T-module then G is irreducible. Furthermore,*

(i) *in each of the cases $T = V_4$, $T = C$, and $T = A_4$, G is irreducible only if M is faithful;*

(ii) *in the case $T = D$, G is irreducible only if either M is faithful, or G is conjugate to a subgroup of BV_4;*

(iii) *in the case $T = S_4$, G is irreducible only if either M is faithful, or $\mathsf{Z}(G) = M$ and $|M|$ is even.*

Proof. The obvious generalisation of [5, Proposition 1.3.4] yields that if M is faithful then G has an irreducible subgroup with diagonal subgroup M and projection group C or V_4. This takes care of the first assertion. For (i), note that if M is not faithful then by Lemma 2.1 either G has a normal abelian subgroup of index less than 4, or $|G : \mathsf{Z}(G)| < 16$. But then G is reducible, by [7, p. 30, Exercise 2.9(b)] or [7, p. 28, (2.30)]. Lemma 2.1(ii) and Proposition 4.1 give (ii). Let $T = S_4$ and suppose M is not faithful, so that M is scalar. If $|M|$ is odd then by Proposition 3.8(i) and Corollary 3.5 there is a conjugate of G in XS_4, which of course leaves invariant the subspace of $\mathbb{C}^{(4)}$ spanned by $(1,1,1,1)^{\mathsf{T}}$. ∎

5. THE CONJUGACY PROBLEM

Recall that two finite T-submodules of B are isomorphic if and only if they are equal. This fact is used in proving the first result of this section.

PROPOSITION 5.1. *Let G and H be finite subgroups of $M(4)$ such that $\pi G = \pi H = T$, and let M be a T-submodule of $B \cap G$. If θ is an isomorphism of G onto H such that $\theta(B \cap G) = B \cap H$, then $M \sim_{\mathsf{N}_{S_4}(T)} \theta(M)$.*

Proof. Cf. the proof of [5, Theorem 2.7]. ∎

Remark 5.2. Suppose G, H, M are as in Proposition 5.1, and θ is any isomorphism from G to H such that $\theta(M) \leq B \cap H$. Then M is a faithful T-module if and only if $\theta(M)$ is also faithful. In that situation, $\theta(B \cap G) = B \cap H$.

PROPOSITION 5.3. *If M is a finite faithful T-submodule of B then* $\mathsf{N}_{GL(4)}(M) \leq M(4)$.

Proof. Cf. the proof of [5, Proposition 1.3.7]. ∎

The following is used frequently in Section 6.

PROPOSITION 5.4. *Let G and H be finite subgroups of $M(4)$, where $\pi G = \pi H = T$, and suppose that $G^m = H$ for some $m \in GL(4)$. Let ϖ be the set of prime divisors of $|T|$.*

(i) *$(G \cap B_\varpi T)^m \sim_B H \cap B_\varpi T$ and $(B_{\varpi'} \cap G)^m = B_{\varpi'} \cap H$.*

(ii) *If M is a faithful T-submodule of $B \cap G$ such that $M^m \leq B \cap H$ then $m \in D(4)\mathsf{N}_{S_4}(T)$.*

Proof. All subgroups of H of order $|H \cap B_\varpi T|$ are conjugate, and $B_{\varpi'} \cap H$ is the unique subgroup of H whose order is a ϖ'-number, so (i) is clear. Proposition 5.1 and Remark 5.2 imply that $M^m = M^s$ for some $s \in \mathsf{N}_{S_4}(T)$. By Proposition 5.3, $ms^{-1} \in M(4)$, so that $m \in M(4)$. Now (ii) follows from (i). ∎

Remark 5.5. Rather obviously, we can sometimes strengthen the second part of Proposition 5.4(i). Say M and N are finite subgroups of $B_{2'}$ such that either both are S_4-modules, or both are non-faithful D-modules. If they are $GL(4)$-conjugate then $M \cap X_p = N \cap X_p$ and $|M_p| = |N_p|$ for each p; but this is enough to guarantee $M = N$.

Only torsion elements of $M(4)$ figure in the conjugacy problem.

LEMMA 5.6. *Let ϖ be a set of primes and suppose G, H are finite subgroups of $B_\varpi T$ such that $\pi G = \pi H = T$. Then $G \sim_{M(4)} H$ if and only if $G \sim_{B_\varpi S_4} H$.*

Proof. Cf. [5, Remark 1.3.8]. ∎

6. THE SUBLISTS FOR T A SUBGROUP OF D

Let T stand for one of V_4, C, or D. Our goal in this section is to produce three separate lists of finite irreducible subgroups of BD, such that a given finite irreducible subgroup G of $D(4)T$ is conjugate to a single group in the union of these three lists.

By Theorem 3.11(iii), we may assume $G = G_2 N$, where $G_2 = G \cap B_2 D$ and $N = G \cap B_{2'}$. Essentially, we have almost all possibilities for G_2 from [5], and all possibilities for N from Section 2. To save repetition, this notation G, G_2, and N, with respect to the relevant T, is fixed at the beginning of each of the next three subsections.

6.1. $T = V_4$

We record here some errata pertaining to [5, Theorem 3.3.1] and also to the dependent [5, Theorems 3.3.14 and 6.1.1]. The parameter ranges for the groups

$$\left\langle ax_{i+1}^{\varepsilon} y_{j+1}^{\eta}, bx_{i+1}^{\gamma(1-\varepsilon)}, F(i,j,k,k,0,1)\right\rangle, \qquad \left\langle ax_{i+1}^{\varepsilon}, b, F(i,j,k,l,1,1)\right\rangle$$

(written in the notation of [5]) are incorrect. Specifically, k should be restricted in both instances to $k \geq 1$: this achieves the irreducibility claimed in [5, Theorem 3.3.1]. The amended versions of this theorem and dependent results are referred to with a superscript *.

As it stands, [5, Theorem 3.3.1] is a complete and irredundant list of $B_2 S_4$-conjugacy class representatives of the finite groups in $B_2 V_4$ with noncyclic diagonal subgroup and projection group V_4. Split extensions by these groups, and by groups with cyclic diagonal subgroup as given in the next result, appear in the list of this subsection.

PROPOSITION 6.1.1. (i) *Extend the list of* [5, *Theorem* 3.3.1] *by adding* $\langle a, b \rangle$;

$$\left.\begin{array}{l} \langle a, bu_{1,2}^{\eta}, M(i,-1,-1,-1)\rangle \\ \langle ax_{i+1,2}^{\varepsilon}, bx_{i+1,2}^{1-\varepsilon} u_{1,2}^{\varepsilon}, M(i,-1,-1,-1)\rangle \end{array}\right\} \quad i \geq 0;$$

$$\left.\begin{array}{l} \langle ax_{1,2}^{\varepsilon} y_{i+1,2}^{\eta}, bx_{1,2}^{\mu(1-\varepsilon)}, M(-1,i,-1,-1)\rangle \\ \langle ax_{i+1,2}^{\varepsilon}, b, M(i+1,1,-1,-1)\rangle \\ \langle ax_{1,2}^{\varepsilon}, b, M(1,i+1,-1,-1)\rangle \\ \langle ax_{1,2}^{\varepsilon}, b, M(-1,i+1,1,-1)\rangle \end{array}\right\} \quad i \geq 1;$$

where ε, η, μ *range over* $\{0,1\}$. *Then a finite subgroup of* $B_2 V_4$ *with projection group* V_4 *is* $M(4)$*-conjugate to one and only one group in the extended list.*

(ii) *Extend the list of* [5, *Theorem* 3.3.1*] *by adding the irreducible groups*

$$\langle ax_{1,2}^{\varepsilon}, b, M(-1,i+1,1,-1)\rangle, \qquad i \geq 1,\ \varepsilon \in \{0,1\}.$$

Then a finite irreducible subgroup of B_2V_4 *is* $M(4)$*-conjugate to one and only one group in the extended list*.

Proof. Let H be a finite subgroup of B_2V_4 with projection group V_4. Using Lemma 5.6, we see that if $B_2 \cap H$ is noncyclic then H is $M(4)$-conjugate to a single group listed in [5, Theorem 3.3.1]. Moreover, if H is irreducible then H is $M(4)$-conjugate to a single group listed in [5, Theorem 3.3.1*]. From now on we assume $B_2 \cap H$ is one of the finite cyclic V_4-submodules M of B_2 as given by (1) and (2)—so H is not $M(4)$-conjugate to any group in either unextended list. With a view to an intended application, we take the d-conjugates of the V_4-modules listed in (1). Since, at each order, the V_4-modules in (2) are all S_4-conjugate, just one of them needs to be considered here.

If $M \neq 1$ is scalar then $|H^2(V_4, M)| = 8$ by the universal coefficient theorem. Otherwise, we calculate $|H^2(V_4, M)|$ by applying [5, (3.13) and Propositions 3.2.1–3.2.3]. If $M = M(-1, i, -1, -1)$ then this order is 8; in the other cases, it is 2. By Theorem 3.9, and the same sort of manipulations as in the proof of [5, Theorem 3.2.9], it may then be seen that H is B_2-conjugate to precisely one of $|H^2(V_4, M)|$ extensions of M by V_4 in B_2V_4, for each M. These extensions may be sorted into $M(4)$-conjugacy classes, using essentially the method outlined at the beginning of the proof of [5, Theorem 3.3.1] (cf. also the discussion after this lemma). By Theorem 4.2(i), H is irreducible if and only if $B_2 \cap H$ is a faithful V_4-module. We extend the lists of [5, Theorems 3.3.1 and 3.3.1*] according to the procedure and criterion just detailed. This proves the proposition. ∎

Our solution of the conjugacy problem in this subsection utilises the information given in Subsection 2.3 about orbits of finite V_4-submodules of B_2, under action by S_4 and its Sylow 2-subgroups. We also need some associated information about the $M(4)$-normalisers of finite subgroups of B_2V_4. That is, for each group G_2 in the list of Proposition 6.1.1(i) (which contains the list of Proposition 6.1.1(ii)), we need to find $\pi \mathsf{N}_{M(4)}(G_2)$. This is tedious but not difficult to do; we revisit the proof of [5, Theorem 3.3.1]. But first we show that $\pi \mathsf{N}_{M(4)}(G_2) \geq V_4$. Let H be any finite subgroup of B_2V_4 such that $B_2 \cap H = B_2 \cap G_2$ and $\pi H = V_4$. If $B_2 \cap H$ is noncyclic then it is implicit in the proof of [5, Theorem 3.2.9] that H is actually B_2-conjugate to a group in the list of that theorem (which contains the list of [5, Theorem 3.3.1]). Of course, the relevant group is G_2 if $G_2 \sim_{V_4} H$. Likewise, when $B_2 \cap G_2$ is cyclic, it is implicit in the proof of Proposition 6.1.1 that $G_2^t \sim_{B_2} G_2$ for all $t \in V_4$.

Suppose $B_2 \cap G_2$ is noncyclic. Our determination of $\pi \mathsf{N}_{M(4)}(G_2)$ is aided by the S_4-orbit display in the proof of [5, Proposition 3.1.12]. If $B_2 \cap G_2 = F(i, j, k, l, \delta, \xi, \alpha)$ is in the seventh orbit or one of the first five orbits, or is $F(i, j, 0, 0, 0, 1, 1)$, then $\pi \mathsf{N}_{M(4)}(G_2) \leq \mathsf{N}_{S_4}(B_2 \cap G_2) \leq V_4\langle e \rangle$;

if $B_2 \cap G_2 = F(i, j, k, l, 1, 0, 1)$ then $\pi \mathsf{N}_{M(4)}(G_2) \leq V_4\langle e\rangle^d$; otherwise, $\pi \mathsf{N}_{M(4)}(G_2)$ could be any one of the subgroups of S_4 containing V_4. Which one is the case depends on the order parameters j, k, l and the generators of G_2 not in $B_2 \cap G_2$. If $\mathsf{N}_{S_4}(B_2 \cap G_2) \neq V_4$ then by Propositions 2.2.1 and 2.3.1, $B_2 \cap G_2$ is normalised by $d^\sigma e$ for a single value of σ or all three possible values. Since we know how $\langle d, e\rangle$ acts as S_3 on j, k, l in the label of $B_2 \cap G_2$, it is straightforward to work out the value or values of σ. When $B_2 \cap G_2$ is normalised by $d^\sigma e$ for a single value of σ, we only have to locate the unique B_2-conjugate of $G_2^{d^\sigma e}$ in the list of [5, Theorem 3.2.9]. If this is G_2 then $\pi \mathsf{N}_{M(4)}(G_2) = V_4\langle e\rangle^{d^\sigma}$; otherwise, $\pi \mathsf{N}_{M(4)}(G_2) = V_4$. When $B_2 \cap G_2$ is an S_4-module, either two or at most three calculations are required to establish that $\pi \mathsf{N}_{M(4)}(G_2) = S_4$ or $\pi \mathsf{N}_{M(4)}(G_2) \leq V_4\langle e\rangle^{d^\sigma}$ for some σ. If $\pi \mathsf{N}_{M(4)}(G_2) \neq V_4\langle e\rangle^{d^\sigma}$ for all σ then it may be observed that $d \notin \pi \mathsf{N}_{M(4)}(G_2)$, so that $\pi \mathsf{N}_{M(4)}(G_2) = V_4$.

As an example, let

$$G_2 = \left\langle ax^{\varepsilon}_{i+1,2}y^{\eta}_{j+1,2}, bx^{\gamma}_{i+1,2}u^{\mu}_{k+1,2}v^{\nu}_{l+1,2}, F(i,j,k,l,0,0)\right\rangle.$$

If $j < k < l$ then $\pi \mathsf{N}_{M(4)}(G_2) = V_4$. If $j = k = l$ then $\pi \mathsf{N}_{M(4)}(G_2)$ is S_4 for $\varepsilon = \eta = \gamma = \mu = \nu = 0$, and $V_4\langle e\rangle^{d^2}$ for $\varepsilon = \mu = 1$ and $\eta = \gamma = \nu = 0$. If $j = k \neq l$ and $\varepsilon = \gamma = \nu = 0$, $\eta = \mu = 1$, or $\varepsilon = \eta = \gamma = \mu = 0$, $\nu = 1$, then $\pi \mathsf{N}_{M(4)}(G_2) = V_4\langle e\rangle^d$.

If $B_2 \cap G_2$ is $M(i, -1, -1, -1)$ or $M(-1, i+1, 1, -1)$ then $\mathsf{N}_{S_4}(B_2 \cap G_2)$ is S_4 or V_4, respectively, whereas $\mathsf{N}_{S_4}(B_2 \cap G_2) = V_4\langle e\rangle$ for all other cyclic $B_2 \cap G_2$. We proceed as above to find $\pi \mathsf{N}_{M(4)}(G_2)$. Again, it may be observed empirically that when this group contains A_4, it is all of S_4. We state this property formally.

LEMMA 6.1.2. *If G_2 is in the (extended) list defined in Proposition* 6.1.1(i), *then $\pi \mathsf{N}_{M(4)}(G_2)$ is one of V_4, $V_4\langle e\rangle^{d^\sigma}$, or S_4, where* $0 \leq \sigma \leq 2$.

To finish our preparation for the main result of this subsection, we define below lists $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$, of finite irreducible subgroups of BV_4. In the definition of $\mathscr{F}_i$, a typical element G_2N of the list is described by giving the possibilities for G_2 and N. Note that listed groups are irreducible by Theorem 4.2.

$\mathscr{F}_1$: G_2 ranges over the groups listed in [5, Theorem 3.3.14*], and N ranges over the finite scalar V_4-submodules of $B_{2'}$.

$\mathscr{F}_2$: G_2 ranges over the groups in the extended list defined in Proposition 6.1.1(i), and N ranges over

the faithful finite V_4-submodules of $B_{2'}$, if $\pi \mathsf{N}_{M(4)}(G_2) = V_4$;

the relevant V_4-submodules of $B_{2'}$ listed in Theorem 2.3.2(ii) that are faithful, if $\pi \mathsf{N}_{M(4)}(G_2) = V_4\langle e\rangle^{d^\sigma}$ for some σ;

the V_4-submodules of $B_{2'}$ listed in Theorem 2.3.2(i) that are faithful, if $\pi \mathsf{N}_{M(4)}(G_2) = S_4$.

$\mathscr{F}_3$: G_2 ranges over the groups in the extended list defined in Proposition 6.1.1(ii), and N ranges over

all finite type 2^{d^σ} modules $\prod_p F(i_p, j_p, -1, -1; p)^{d^\sigma}$, $0 \le \sigma \le 2$, if $\pi \mathsf{N}_{M(4)}(G_2) = V_4$;

all finite type 2^{d^τ} modules $\prod_p F(i_p, j_p, -1, -1; p)^{d^\tau}$, where $\tau = \sigma$ or $\tau = \sigma + 1$, if $\pi \mathsf{N}_{M(4)}(G_2) = V_4 \langle e \rangle^{d^\sigma}$ for some σ;

the finite type 2 modules $\prod_p F(i_p, j_p, -1, -1; p)$, if $\pi \mathsf{N}_{M(4)}(G_2) = S_4$.

Remark 6.1.3. The following is to be read in conjunction with the definition of $\mathscr{F}_2$. If N is type 1 then $N_p = F(i_p, j_p, j_p, j_p; p)$ for some p and $j_p \ge 0$. If N is type 2^{d^σ} then for some p, either $N_p = F(i_p, j_p, j_p, j_p; p)$, $j_p \ge 0$; or $N_p = F(i_p, j_p, k_p, k_p; p)^{d^\sigma}$, $k_p \ge 0$. If N is type 3^{d^σ} then there is no restriction on N.

We define a sublist $\mathscr{F}_3'$ of $\mathscr{F}_3$ as follows: $G_2 N \in \mathscr{F}_3'$ if and only if G_2 is in the list of [5, Theorem 3.3.14*] when $\mathsf{C}_{V_4}(N) = \langle a \rangle$ (that is, N is a type 2 V_4-submodule $\prod_p F(i_p, j_p, -1, -1; p)$ of $B_{2'}$), and in the list of Proposition 6.1.1(ii) when $\mathsf{C}_{V_4}(N)$ is $\langle b \rangle$ or $\langle ab \rangle$. Denote the (disjoint) union $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3'$ by $\mathscr{F}$.

THEOREM 6.1.4. *A finite irreducible subgroup G of $D(4)V_4$ is conjugate to one and only one group in $\mathscr{F}$.*

Proof. By [5, Theorem 3.3.14*], Theorem 2.3.2 (and the comments just before it), Proposition 6.1.1 and Lemma 6.1.2, $G = G_2 N$ is conjugate to a group in $\mathscr{F}_1$, $\mathscr{F}_2$, or $\mathscr{F}_3$, according as $\mathsf{C}_{V_4}(N) = V_4$, $\mathsf{C}_{V_4}(N) = 1$, or $|\mathsf{C}_{V_4}(N)| = 2$, respectively. So we take $G \in \bigcup_i \mathscr{F}_i$. Suppose G and $K \in \bigcup_i \mathscr{F}_i$ are conjugate. Write $K = K_2 M$, where $K_2 = K \cap B_2 V_4$ and M is a finite V_4-submodule of $B_{2'}$. By Proposition 5.4(i), we have $G_2^m = K_2$ and $N^m = M$ for some $m \in GL(4)$.

If $G \in \mathscr{F}_i$ then $|\mathsf{C}_K(M)| = |\mathsf{C}_G(N)|$ is $|G|$, $|G|/4$, or $|G|/2$, as $i = 1, 2$, or 3, respectively. Hence G and K both lie in $\mathscr{F}_i$ for some i.

Distinct groups in $\bigcup_i \mathscr{F}_i$ are not $M(4)$-conjugate. For if $m \in M(4)$ then $G_2 = K_2$ by the very definition of the $\mathscr{F}_i$, Proposition 6.1.1 and [5, Theorem 3.3.14*]. Therefore, N and M are $\pi \mathsf{N}_{M(4)}(G_2)$-conjugate. By Theorem 2.3.2 and the definition of the $\mathscr{F}_i$ again, this implies $N = M$.

Distinct groups in $\mathscr{F}_1$ are certainly not conjugate. If $G \in \mathscr{F}_2$ then N is a faithful V_4-module, so that $m \in M(4)$ by Proposition 5.4(ii), and $G = K$ by the preceding paragraph. Therefore, distinct groups in $\mathscr{F}_2$ are not conjugate. So the only groups in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ that could be conjugate are those in $\mathscr{F}_3$. We now determine all conjugacy between groups in $\mathscr{F}_3$ and thereby establish the theorem. (Much of the proof of [5, Theorem 3.3.10] resurfaces in what follows, and this proof should be consulted for assertions not explicitly justified.)

Assume $G \in \mathscr{F}_3$. Let $f \in GL(4)$ be as defined in [5, Proposition 3.3.9]; recall that f normalises each subgroup of XY. Suppose G_2 is in the list of [5, Theorem 3.3.1*]. If G_2 is not in the list of [5, Theorem 3.3.14*], then G_2^{zf} is B_2S_4-conjugate to a group in the list of [5, Theorem 3.3.14*], for some $z \in \langle u_{2,2} \rangle$. Since zf normalises N such that $\mathsf{C}_{V_4}(N) = \langle a \rangle$, it follows that $G \in \mathscr{F}_3'$ or $G^{zf} \in \mathscr{F}_3'$ here. If $G_2 = \langle ax_{1,2}^{\varepsilon}, b, M(-1, i+1, 1, -1) \rangle$ then, using [5, (3.22)–(3.24)], we see that G_2^f is B_2S_4-conjugate to a group in the list of [5, Theorem 3.3.1*] with diagonal subgroup $F(0,0,1,i,1,0)$. If also $\mathsf{C}_{V_4}(N) = \langle a \rangle$ then G is conjugate to a group in $\mathscr{F}_3'$, by the previous part. Each group in $\mathscr{F}_3$ is thus conjugate to at least one group in $\mathscr{F}_3'$. The remainder of the proof is a verification that distinct groups in $\mathscr{F}_3'$ are not conjugate.

We assume now that $G, K \in \mathscr{F}_3'$ and $m \notin M(4)$. The concepts of types I and II linear isomorphism defined in [5, p. 47] may be generalised to apply to all finite irreducible subgroups of $D(4)V_4$. If there were a type II linear isomorphism from G to K then N and M would be scalar. So m induces a type I linear isomorphism from G to K. By restriction (and up to M-conjugacy), m therefore induces a type I linear isomorphism from G_2 to K_2. That is, $B_2 \cap G_2$ has a maximal V_4-submodule L such that that $\mathsf{C}_{V_4}(LN) = \langle t \rangle$ for some involution t of V_4, and L^mM is a maximal V_4-submodule of $B \cap K$. We have a factorisation of m as z_1fz_2, for some $z_1 \in S_4\langle u_{2,2} \rangle$, $z_2 \in M(4)$, and f as above.

Let $t = a$, so that $z_1 \in \langle b, u_{2,2} \rangle$ and G_2 is in the list of [5, Theorem 3.3.14*]. We know in advance, then, that $G_2 = K_2$. In fact, $G_2^{z_1f}$ is B_2-conjugate either to G_2 or to a group H in the list of Proposition 6.1.1(ii) but not in the list of [5, Theorem 3.3.14*]. Therefore $K \sim_{M(4)} G$ or $K \sim_{M(4)} HN$. The latter possibility leads to $K = HN \notin \mathscr{F}_3'$, forcing $K = G$.

Henceforth $t \neq a$, and so $B_2 \cap G_2$ is noncyclic. Suppose L is noncyclic. Then $B_2 \cap G_2$ must be $F(0,0,0,0,1,1,1)$ or one of the $F(i,1,1,1,0,0)$, $i \geq 1$. Of the groups with diagonal subgroup $F(0,0,0,0,1,1,1)$ in the list of [5, Theorem 3.3.1*], only the split extension is involved in a type I isomorphism. This extension is normalised by S_4, meaning that N should be centralised by a, according to the definition of $\mathscr{F}_3$. But $\mathsf{C}_{V_4}(N) \neq \langle a \rangle$. When $B_2 \cap G_2 = F(i,1,1,1,0,0)$, the choices for G_2 are dictated by [5, Theorem 3.3.1* and (3.25)–(3.27)]. For $t = b$, these choices are

$$\langle a, b, F(i,1,1,1,0,0) \rangle, \quad \langle ax_{i+1,2}, bu_{2,2}^{\mu}, F(i,1,1,1,0,0) \rangle, \quad \mu \in \{0,1\};$$

and for $t = ab$ they are

$$\langle a, bv_{2,2}^{\nu}, F(i,1,1,1,0,0)\rangle, \quad \langle ax_{i+1,2}, b, F(i,1,1,1,0,0)\rangle, \quad \nu \in \{0,1\}.$$

Calculating $\pi \mathsf{N}_{M(4)}(G_2)$, and comparing this with $\mathsf{C}_{V_4}(N) = \langle t\rangle$ and the definition of $\mathscr{F}_3$, we reduce the possibilities for G_2 to the non-split extensions if $t = b$, and $\langle a, bv_{2,2}, F(i,1,1,1,0,0)\rangle$ if $t = ab$. If $t = b$ then $(G_2 N)^{z_1 f} = (G_2 N)^{def}$ is $(G_2 N)^{de}$, or is in $\mathscr{F}_3$ but not in $\mathscr{F}_3'$, so cannot be $M(4)$-conjugate to anything in $\mathscr{F}_3'$. Hence $K = G$ to avoid a contradiction. The case $t = ab$ is dealt with analogously.

Suppose L is cyclic. Then $G_2 = K_2$ and $B_2 \cap G_2 = B_2 \cap K_2$ is one of the $F(i,1,0,0,0,1,1)$, $F(0,0,i,1,1,0,1)$, or $F(0,0,1,i,1,0)$. In the first and second cases, G_2 must split over $B_2 \cap G_2$, and there is a single choice for N: in the first case because $\pi \mathsf{N}_{M(4)}(G_2) = V_4\langle e\rangle$, and in the second because L is the unique maximal V_4-submodule of $B_2 \cap G_2$ with $\mathsf{C}_{V_4}(L) = \langle b\rangle$ or $\langle ab\rangle$. Just the case $B_2 \cap G_2 = F(0,0,1,i,1,0)$ is left to concern us. This may be handled using uniqueness of L (for $i > 1$) or a combination of ideas seen already (for $i = 1$). The proof is complete. ∎

6.2. $T = C$

We obtain tremendous simplifications in this subsection over the previous one: unlike V_4, C has a single involution and reasonably small S_4-normaliser.

LEMMA 6.2.1. *G is conjugate to a subgroup of $D(4)V_4$ if and only if $B \cap G$ has a C-submodule of index 2 (containing N) with centraliser $\langle b\rangle$ in C.*

Proof. With regard to (1), (3), and using Proposition 4.1, one carries over the proof of [5, Proposition 4.2]. ∎

PROPOSITION 6.2.2. *Extend the list of* [5, *Theorem* 4.8] *by adding the following groups.*

For $i \geq -1$,

$$\langle cx_{i+2,2}^{\varepsilon}, M(i,-1,-1,-1)\rangle;$$

for $i \geq 1$,

$$\langle cx_{2,2}^{\eta}, M(-1,-1,i,-1)\rangle,$$

$$\langle cx_{i+2,2}^{\eta}, M(i+1,-1,1,-1)\rangle,$$

$$\langle cx_{2,2}^{\eta}, M(1,-1,i+1,-1)\rangle,$$

$$\langle c, M(i+2,1,2,-1)\rangle,$$

$$\langle c, M(2,1,i+2,-1)\rangle;$$

for $i, j \geq 1$,

$$\langle cx^{\varepsilon}_{i+2,2}, C(i, j, 1, 0, 0)\rangle,$$

$$\langle cx^{\varepsilon}_{i+2,2}, C(i, j, 0, 0, 1)\rangle,$$

$$\langle cx^{\varepsilon}_{i+2,2}, C(i, j, 0, 0, 1, 0)\rangle,$$

$$\langle cx^{\eta}_{i+2,2}, C(i, j, 0, 0, 1, \xi)\rangle;$$

for $i = j = 0$ or $i, j \geq 1$,

$$\langle cx^{\eta}_{i+2,2}, C(i, j, 1, 1, 0)\rangle,$$

$$\langle cx^{\eta}_{i+2,2}, C(i, j, 0, 1, 1)\rangle,$$

$$\langle cx^{\eta}_{i+2,2}, C(i, j, 0, 1, 1, 1)\rangle;$$

where ε, η, *and* ξ *range, respectively, over* $\{0, 1, 2\}$, $\{0, 1\}$, *and* $\{-1, 1\}$. *This extended list is a complete and irredundant list of representatives for the* $D(4)D$*-conjugacy classes of finite subgroups of* B_2C *with projection group* C.

Proof. Cf. the proof of Proposition 6.1.1 (the second cohomology calculations, for coefficient modules as specified in (1), (3), and Subsection 2.2, are far easier here). The reducible and other groups omitted from the initial list constructed in the proof of [5, Theorem 4.8] are here restored. ∎

Lists $\mathscr{C}_1$ and $\mathscr{C}_2$ of finite irreducible subgroups of $D(4)C$ are defined below, by describing a typical element G_2N in each list.

$\mathscr{C}_1$: G_2 ranges over the groups listed in [5, Theorem 4.8], and N ranges over the finite D-submodules $\prod_p F(i_p, -1, k_p, -1; p)$ of $B_{2'}$.

$\mathscr{C}_2$: G_2 ranges over the groups in the extended list defined in Proposition 6.2.2, and N ranges over

the finite faithful C-submodules of $B_{2'}$, if $\pi \mathsf{N}_{M(4)}(G_2) = C$;

the faithful C-modules in the list of Corollary 2.3.5, if $\pi \mathsf{N}_{M(4)}(G_2) = D$.

A C-submodule N of $B_{2'}$ is faithful if and only if N_p is faithful for at least one odd prime p. If $N_p = F(i, j, k, j; p)$ then this is equivalent to $j \geq 0$; whereas $N_p = C(i, j, k, r, \xi; p)$ is faithful for all allowed values of parameters. The determination of $\pi \mathsf{N}_{M(4)}(G_2)$, given G_2, is equally straightforward. Since all groups listed in Proposition 6.2.2 and [5, Theorem 4.8] are of the form $G_2 = \langle cx^{\alpha}_{i+2,2}, B_2 \cap G_2\rangle$, where $|G_2 \cap X_2| = 2^{i+1}$ and $\alpha \in \{0, 1, 2\}$, it is evident that $C \leq \pi \mathsf{N}_{M(4)}(G_2) \leq D$. If α is 0 or 2 then $\pi \mathsf{N}_{M(4)}(G_2) = D$. If $\alpha = 1$ then by [5, Lemma 4.7], $\pi \mathsf{N}_{M(4)}(G_2) = D$ if and only if $x_{i+1,2} \in (B_2 \cap G_2)YUV \cap X$.

Denote $\mathscr{C}_1 \cup \mathscr{C}_2$ by $\mathscr{C}$.

THEOREM 6.2.3. *Let G be a finite irreducible subgroup of $D(4)C$. Then G is conjugate to a group in $\mathscr{F} \cup \mathscr{C}$. Distinct groups in $\mathscr{C}$ are not conjugate, and no group in $\mathscr{C}$ is conjugate to a group in $\mathscr{F}$.*

Proof. Either $\mathsf{C}_C(N) \geq \langle b \rangle$, or N is a faithful C-module. In the former case, G_2 is irreducible, and either G is conjugate to a group in $\mathscr{F}$, by Lemma 6.2.1 and Theorem 6.1.4, or G_2 is B_2D-conjugate to a group in the list of [5, Theorem 4.8]—but then G is B_2D-conjugate to a group in $\mathscr{C}_1$. By Proposition 6.2.2, G_2 is $D(4)D$-conjugate to a group in the list of that proposition. So if N is faithful then G is $D(4)D$-conjugate to a group in $\mathscr{C}_2$.

A group in $\mathscr{C}_1$ is not conjugate to a group in $\mathscr{C}_2$. Furthermore, a group in $\mathscr{C}_1$ is not conjugate to a group in $\mathscr{F}$: otherwise G_2 would be conjugate to a subgroup of B_2V_4, which contradicts [5, Theorem 4.8]. It then follows from Lemma 6.2.1 that a group in $\mathscr{C}_1$ has a unique self-centralising normal subgroup of index 4. Hence conjugate groups in $\mathscr{C}_1$ are equal, by Proposition 5.4 and [5, Theorem 4.8].

By Proposition 5.4, Corollary 2.3.5 and Proposition 6.2.2, distinct groups in $\mathscr{C}_2$ are not conjugate. Finally, we realise that no group in $\mathscr{C}_2$ is conjugate to a group in $\mathscr{F}$, by Lemma 6.2.1. ∎

6.3. $T = D$

LEMMA 6.3.1. *Suppose $B \cap G$ is a faithful D-module. Then G is conjugate to a group in $\mathscr{F}$ or $\mathscr{C}$ if and only if G has an abelian normal subgroup H of index 4 (containing N) such that $\pi H = \langle ac, b \rangle$.*

Proof. One direction is a consequence of Proposition 4.1. For the other, set $M = B \cap G$ and note that G has an abelian normal subgroup H of index 4 such that $|M : H \cap M| = 2$. Hence $|\pi H| = 4$ and $N \leq H$. Either M_2 or N is faithful, by Lemma 2.1(ii). In the former case $M_2 \cap H$ cannot be scalar. Thus, in both cases $\pi H \neq C$ and $\pi H \neq V_4$. ∎

PROPOSITION 6.3.2. *Extend the list of* [5, *Theorem* 5.7] *by adding the groups* $\langle ax^{\varepsilon}_{0,2}, c \rangle$;

$$\left\langle ax^{\varepsilon}_{i+1,2}(u_{2,2}v_{1,2})^{\eta}, cx^{\nu}_{i+1,2}x^{\eta}_{2,2}, M(i,-1,-1,-1) \right\rangle, \qquad i \geq 0;$$

$$\left. \begin{array}{l} \left\langle ax^{\varepsilon}_{1,2}u^{\eta}_{i+1,2}v^{\nu}_{1,2}, cx^{\nu}_{2,2}, M(-1,-1,i,-1) \right\rangle \\ \left\langle au^{\varepsilon}_{2,2}v^{\varepsilon-\eta}_{1,2}, cx^{\eta}_{i+2,2}, M(i+1,-1,1,-1) \right\rangle \\ \left\langle au^{\varepsilon}_{i+2,2}v^{\varepsilon-\eta}_{1,2}, cx^{\eta}_{2,2}, M(1,-1,i+1,-1) \right\rangle \end{array} \right\} \quad i \geq 1;$$

$$\left\langle ax^{\varepsilon}_{i+1,2}u^{\eta}_{j+1,2}y^{\mu}_{1,2}, cx^{\nu}_{i+1,2}, C(i,j,0,0,1) \right\rangle, \qquad i,j \geq 1;$$

$$\left\langle ax_{i+1,2}^{\varepsilon}y_{1,2}^{\eta}, c(x_{i+2,2}u_{j+2,2})^{\mu}, C(i,j,0,1,1)\right\rangle, \quad i=j=0 \quad or \quad i,j \geq 1;$$

where $\varepsilon, \eta, \mu, \nu$ *range, independently, over* $\{0,1\}$. *A finite subgroup of* $B_2 D$ *with projection group* D *is* $D(4)D$*-conjugate to precisely one of the groups in this extended list.*

Proof. Cf. the proof of Proposition 6.1.1. Groups with non-faithful diagonal subgroup, omitted from the list of [5, Theorem 5.7], are here included. Otherwise, we use Theorem 3.9 and techniques like those used to prove [5, Theorem 5.7]. ∎

We define lists $\mathscr{D}_1$ and $\mathscr{D}_2$ of finite irreducible subgroups of $D(4)D$ with the following description of elements $G_2 N$ in each list (recall Corollary 2.3.4).

$\mathscr{D}_1$: G_2 ranges over the groups in the modified list of [5, Theorem 5.12], and N ranges over the finite D-submodules $\prod_p F(i_p, -1, k_p, -1; p)$ of $B_{2'}$.

$\mathscr{D}_2$: G_2 ranges over the groups in the extended list defined in Proposition 6.3.2, and N ranges over the finite faithful D-submodules of $B_{2'}$.

A finite D-submodule N of $B_{2'}$ is faithful if and only if, for at least one odd prime p, $N_p = F(i,j,k,j;p)$ with $j \geq 0$.

Denote $\mathscr{D}_1 \cup \mathscr{D}_2$ by $\mathscr{D}$.

THEOREM 6.3.3. *A finite irreducible subgroup of* $D(4)D$ *with projection group* D *is conjugate either to a group in* $\mathscr{F} \cup \mathscr{C}$, *or to one and only one group in* $\mathscr{D}$. *No group in* $\mathscr{D}$ *is conjugate to a group in* $\mathscr{F} \cup \mathscr{C}$.

Proof. Suppose first that N is not a faithful D-module. If $B_2 \cap G_2$ is not faithful then by Theorem 4.2(ii), G is conjugate to a subgroup of BV_4. If $B_2 \cap G_2$ is faithful then we can take G_2 to be in the list of [5, Theorem 5.7]. Either G_2 is in the list of [5, Proposition 5.11], or G_2 is conjugate to a subgroup of $B_2 V_4$ or $B_2 C$; in the latter event G_2 has a (self-centralising) abelian normal subgroup H of index 4 such that $\pi H = \langle ac, b\rangle$, by Lemma 6.3.1. Considering HN in G, we then see that G is conjugate to a group in $\mathscr{F}$ or $\mathscr{C}$. So we assume G_2 is in the list of [5, Proposition 5.11]. Now, by the discussion in [5, pp. 65–66], a matrix that induces a linear isomorphism between distinct groups in the list is of the form $a^{\sigma} w f' z$, where $\sigma \in \{0,1\}$, $w \in YV$, f' is the *de*-conjugate of the matrix f defined in [5, Proposition 3.3.9] and $z \in B_2 D$. In particular, we note that a group in the list of [5, Proposition 5.11] is conjugate to a group in [5, Theorem 5.12] by a matrix that normalises each D-submodule of XU. Therefore, G is conjugate to a group in $\mathscr{D}_1$. Distinct groups in $\mathscr{D}_1$ are not conjugate, by the preceding comments, [5, Theorem 5.12], Proposition 5.4, and Remark 5.5. A group in $\mathscr{D}_1$ is not conjugate to a group in $\mathscr{F}$ or $\mathscr{C}$, since its Sylow 2-subgroup in B_2D is not conjugate to a subgroup of B_2V_4 or B_2C, again by [5, Theorem 5.12].

Suppose N is faithful. Then G is $D(4)D$-conjugate to a group in $\mathscr{D}_2$. Distinct groups in $\mathscr{D}_2$ are not conjugate, by Propositions 5.4 and 6.3.2. Neither can G be conjugate to a group in $\mathscr{F} \cup \mathscr{C}$, by Lemma 6.3.1 and faithfulness of N.

For the usual reason (the maximal $2'$-subgroups have centralisers of different orders), a group in $\mathscr{D}_1$ is not conjugate to a group in $\mathscr{D}_2$. ∎

To sum up: in Subsections 6.1–6.3 we have shown that $\mathscr{F} \cup \mathscr{C} \cup \mathscr{D}$ is a complete and irredundant list of representatives for the GL(4)-conjugacy classes of finite irreducible subgroups of $M(4)$ with projection group in D.

7. THE SUBLIST FOR $T = A_4$

We begin with a technical lemma.

LEMMA 7.1. *Let G and H be finite subgroups of $D(4)A_4$ such that $\pi G = \pi H = A_4$, $GA_4 \cap D(4) \subseteq X_2(G \cap X)YUV$, yet $HA_4YUV \cap X_{2'} \not\subseteq H \cap X$. Then G and H are not conjugate.*

Proof. Since $A_4YUV \leq SL(4)$, by hypothesis $G \leq X_2(G \cap X)SL(4)$. Suppose G and H are conjugate. Then

$$\{\det(h) \mid h \in H\} = \{\det(g) \mid g \in G\} \subseteq \{\det(x) \mid x \in X_2(G \cap X)\}.$$

However, there exist $w \in A_4YUV$ and $h \in H$ such that $hw = \bar{x} \in X_{2'} \setminus (H \cap X)$. So $\det(\bar{x}) = \det(h) = \det(x)$ for some $x \in X_2(G \cap X)$, which implies $\bar{x}x^{-1}$ has order dividing 4. This means that $\bar{x} \in G \cap X = H \cap X$, a contradiction. ∎

THEOREM 7.2. *Define the list $\mathscr{A}$ of finite irreducible subgroups of $D(4)A_4$ to consist of all semidirect products $\hat{G}N$, where the choices for $\hat{G}$ are given below, and N is any finite A_4-submodule of $B_{\{2,3\}'}$ unless stated otherwise. Throughout, $M_3 = F(i', j', j', j'; 3)$ and σ ranges over $\{0, 1\}$.*

For $i', j' \geq -1$ and $i, j \geq 1$,

$$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i, j, j, j, 0, 0), M_3 \rangle,$$

$$\langle ay_{j+1,2}, bu_{j+1,2}v_{j+1,2}, dx^{\sigma}_{i'+1,3}y_{j+2,2}u^{-1}_{j+2,2}, F(i, j, j, j, 0, 0), M_3 \rangle,$$

$$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i, j, j, j, 0, 0, -1, -1), M_3 \rangle,$$

$$\langle ax_{i+1,2}, bu_{j+1,2}, dx^{\sigma}_{i'+1,3}u_{j+2,2}v^{-1}_{j+2,2}, F(i, j, j, j, 0, 0, -1, -1), M_3 \rangle,$$

$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i,j,j,j,0,0,0,-1), M_3\rangle$,

$\left\langle a, bu_{j+1,2}, dx^{\sigma}_{i'+1,3}u_{j+2,2}v^{-1}_{j+2,2}, F(i,j,j,j,0,0,0,-1), M_3\right\rangle$,

$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i,j,j,j,0,1,0), M_3\rangle$.

For $i', j' \geq -1$, and $i = j = 0$ or $i, j \geq 1$,

$$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i,j,j,j,1,1,1), M_3\rangle,$$

$$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i,j,j,j,1,1,2,1), M_3\rangle,$$

$$\langle a, b, dx^{\sigma}_{i'+1,3}, F(i,j,j,j,1,1,2,-1), M_3\rangle.$$

For $j' \geq 0$, or $j' = -1$ and $N_p = F(i'', j'', j'', j''; p)$ with $j'' \geq 0$ for at least one prime $p > 3$,

$$\langle a, b, dx^{\sigma}_{i'+1,3}, M(i,-1,-1,-1), M_3\rangle, \qquad i \geq -1;$$

$$\left\langle ax_{1,2}, bu_{1,2}, dx^{\sigma}_{i'+1,3}u_{2,2}v^{-1}_{2,2}, M(i,-1,-1,-1), M_3\right\rangle, \qquad i \geq 0.$$

A finite irreducible subgroup of $D(4)A_4$ with projection group A_4 is $M(4)$-conjugate to a group in $\mathscr{A}$, and groups in $\mathscr{A}$ are pairwise non-conjugate in $GL(4)$.

Proof. A group in $\mathscr{A}$ has projection group A_4 and diagonal subgroup that is faithful as an A_4-module, so is irreducible by Theorem 4.2.

Let G be a finite subgroup of $M(4)$ such that $\pi G = A_4$. By Theorem 3.11(i), we may assume $G = HK \ltimes N$, where H is the Sylow 2-subgroup of G contained in B_2V_4, K is a Sylow 3-subgroup contained in $D(4)\langle d\rangle$, and $N = G \cap B_{\{2,3\}'}$. Denote the diagonal subgroup of HK by M. (Remember that M and N are both normal in $M(4)$.) By Lemma 2.1(iii), $M_2 = H \cap B_2$ either is cyclic and scalar, or is noncyclic, faithful as an A_4-module, and listed in Proposition 2.2.1. For some $i', j' \geq -1$ we have $M_3 = F(i', j', j', j'; 3)$.

As explained in the opening paragraph of the proof of [5, Theorem 3.2.9], there exist $x_a, x_b \in X_2$, $y_a \in Y_2$, $u_b \in U_2$, and $v_b \in V_2$ such that H is $\langle ax_ay_a, bx_bu_bv_b, M_2\rangle$ (up to B_2-conjugacy, and regardless of whether or not M_2 is cyclic). We write K generically as $\langle dxyuv, M_3\rangle$, for some $x \in X$, $y \in Y$, $u \in U$, and $v \in V$. Since $HM_3 \trianglelefteq HK$, we see that $(ax_ay_a)^{dxyuv} \equiv ax_ay_abx_bu_bv_b \bmod M_2M_3$ and $(bx_bu_bv_b)^{dxyuv} \equiv ax_ay_a \bmod M_2M_3$, so

$$x_by_ay^2u_bu^2v_b^{-1}y_a^d \in M_2M_3, \qquad x_ax_by_au_b^du^2v_b^dv^2 \in M_2M_3. \tag{4}$$

Hence $y^2u^2, u^2v^2 \in B_2M_3$, from which we deduce that $y, u, v \in B_2M_3$. Without loss, then, y, u, v may all be chosen as elements of B_2, and (4) amended accordingly to require inclusion merely in M_2. The condition

$(dxyuv)^3 \in M_3$, satisfied in K, implies

$$(yuv)^{1+d+d^2} \in Y_2U_2V_2 \cap X = \langle x_{1,2} \rangle, \tag{5}$$

and $x^3(yuv)^{1+d+d^2} \in M_3 \cap X$. That is, $x \equiv x^{\sigma}_{i'+1,3}(yuv)^{1+d+d^2} \bmod M_3$ for some $\sigma \in \{0, 1, -1\}$. Conversely, $\langle ax_a y_a, bx_b u_b v_b, dx^{\sigma}_{i'+1,3}(yuv)^{2+d+d^2}, M \rangle$ is an extension of M by A_4 in $B_2 B_3 A_4$, for any finite A_4-submodule M of $B_2 B_3$, as long as (4) and (5) are satisfied and $\langle ax_a y_a, bx_b u_b v_b, M_2 \rangle$ is an extension of M_2 by V_4 in $B_2 V_4$.

Suppose M_2 is noncyclic. Then a B_2-conjugate of H is in the list of [5, Theorem 3.2.9]: the pertinent lines are the first, second, third, eighth, thirteenth, fourteenth, and fifteenth. So we will assume H is somewhere here—as before, this does not affect our assumption on the form of K. Substituting for x_a, y_a, etc. from the list of [5, Theorem 3.2.9], we decide, by (4) and (5), which of the groups in the list can occur as H, and the matching possibilities for y, u, v. In this way, HK is seen to be B_2-conjugate to one of the groups $\hat{G}$ in the statement of the theorem, such that $\hat{G} \cap B_2$ is noncyclic and σ is allowed the extra value -1.

Suppose M_2 is cyclic. Then H is $B_2 S_4$-conjugate to a group in Proposition 6.1.1(i) with diagonal subgroup $M(i, -1, -1, -1)$ for some $i \geq -1$. That is, $H \sim_{B_2} \langle a, b \rangle$ if $i = -1$, or $H \sim_{B_2} \langle ax^{\varepsilon}_{i+1,2}, bx^{\eta}_{i+1,2} u^{\gamma}_{1,2}, M_2 \rangle$ for some $\varepsilon, \eta, \gamma \in \{0, 1\}$, if $i \geq 0$. As above, we use (4) and (5) to find the allowable values of $\varepsilon, \eta, \gamma$, and the corresponding K. Thus we may show that HK is B_2-conjugate to a group $\hat{G}$ listed in this theorem with diagonal subgroup $M = M(i, -1, -1, -1) \times M_3$, where again σ is possibly -1. (For later reference, we point out that the distinct A_4-extensions of each M so derived are not $D(4)A_4$-conjugate. This follows from Theorem 3.9(ii) and the fact that $H^2(A_4, M_2) \cong \mathbb{Z}_2$ for scalar $M_2 \neq 1$, by the universal coefficient theorem.) We dispense with further details of the construction, which are overly familiar by now.

If M_2 is noncyclic then it is faithful as an A_4-module, and G is irreducible. If M_2 is cyclic then $M_3 N$ must be faithful, by Theorem 4.2(i). This is ensured with the stated restrictions on parameter ranges.

So far, we have proved that G is $M(4)$-conjugate to a group in the list that is obtained by replacing the range $\{0, 1\}$ for σ in the definition of $\mathscr{A}$ by the range $\{0, 1, -1\}$. When G is a group in this list with $G \cap B_2 B_3 A_4$ of the form $\langle a, b, dx^{\sigma}_{i'+1,3}, M \rangle$, we have

$$\langle a, b, dx^{-1}_{i'+1,3}, M \rangle^e = \langle a, b, d^{-1} x^{-1}_{i'+1,3}, M \rangle = \langle a, b, dx_{i'+1,3}, M \rangle.$$

This shows that G is $M(4)$-conjugate to a group in $\mathscr{A}$: σ can range over $\{0, 1\}$, as indicated. In fact the same conclusion always holds, although more effort is required to see this when $G \cap B_2 B_3 A_4$ does not have the above form. For example,

$$\begin{aligned}\Big\langle ay_{j+1,2}, bu_{j+1,2}v_{j+1,2}, dx_{i'+1,3}^{-1}y_{j+2,2}u_{j+2,2}^{-1}, F(i,j,j,j,0,0), M_3\Big\rangle^{e} \\ = \Big\langle ay_{j+1,2}, by_{j+1,2}u_{j+1,2}v_{j+1,2}, dx_{i'+1,3}u_{j+2,2}v_{j+2,2}^{-1}, \\ F(i,j,j,j,0,0), M_3\Big\rangle \\ \sim_{y_{j+2,2}^{-1}} \Big\langle ay_{j+1,2}, bu_{j+1,2}v_{j+1,2}, dx_{i'+1,3}y_{j+2,2}^{-1}u_{j+2,2}, \\ F(i,j,j,j,0,0), M_3\Big\rangle \\ \sim_{u_{j+1,2}^{-1}} \Big\langle ay_{j+1,2}, bu_{j+1,2}v_{j+1,2}, dx_{i'+1,3}y_{j+2,2}u_{j+2,2}^{-1}, \\ F(i,j,j,j,0,0), M_3\Big\rangle .\end{aligned}$$

The remaining checks needed to fully prove that G is $M(4)$-conjugate to a group in $\mathscr{A}$ are sufficiently similar to the one just performed to be omitted.

Now we tackle the conjugacy problem. Let $G \in \mathscr{A}$, where $G = \hat{G}N$ in the usual notation, and suppose $G^m \in \mathscr{A}$ for some $m \in GL(4)$. Denote $G^m \cap B_2B_3A_4$ by $\tilde{G}$. By Proposition 5.4(i), $\hat{G}^m \sim_B \tilde{G}$ and $N^m = G^m \cap B_{\{2,3\}'}$. Indeed, $N = N^m$ by Remark 5.5. Since M_3 and $\tilde{G} \cap B_3$ are, respectively, the unique normal subgroups of $\hat{G}$ and $\tilde{G}$ of order $|M_3|$, it follows also from Remark 5.5 that $M_3 = M_3^m$ and therefore $M_3N = G^m \cap B_{2'} = (M_3N)^m$. We show that $\hat{G} = \tilde{G}$, thereby proving distinct groups in $\mathscr{A}$ are not conjugate.

In the case that M_2 is cyclic, M_3N is a faithful A_4-module. Since $m \in \mathsf{N}_{GL(4)}(M_3N)$ and $\hat{G}^m \sim_B \tilde{G}$, we get $\hat{G} \sim_{M(4)} \tilde{G}$ by Proposition 5.4(ii). Now either $\hat{G}^e$ is $\hat{G}$, or $\hat{G}^eN$ is not in $\mathscr{A}$ and is not $D(4)A_4$-conjugate to anything in $\mathscr{A}$, as we expressly pointed out above. Thus $\hat{G} \sim_{D(4)A_4} \tilde{G}$ and so $\hat{G} = \tilde{G}$.

Assume now that M_2 is noncyclic. The Sylow 2-subgroup $\hat{G} \cap B_2V_4$ of G and the Sylow 2-subgroup $\tilde{G} \cap B_2V_4$ of G^m are in the list of [5, Theorem 6.1.1*]. Groups in that list are pairwise non-conjugate. Therefore, $\hat{G} \cap B_2V_4 = \tilde{G} \cap B_2V_4$ and furthermore $B \cap G = B \cap G^m$. Then Lemma 7.1 and inspection of the list in this theorem reveal that $\hat{G} \cap D(4)\langle d\rangle = \tilde{G} \cap D(4)\langle d\rangle$. Hence $\hat{G} = (\hat{G} \cap B_2V_4)(\hat{G} \cap D(4)\langle d\rangle) = (\tilde{G} \cap B_2V_4)(\tilde{G} \cap D(4)\langle d\rangle) = \tilde{G}$, as required. ∎

Remark 7.3. The isomorphism question posed at the start of [5, Chap. 2] has a negative answer for finite irreducible monomial linear groups in general. For

$$\langle a, b, d, F(i,j,j,j,1,1,1)\rangle \cong \langle a, b, dx_{0,3}, F(i,j,j,j,1,1,1)\rangle,$$

although these groups are not conjugate, by Theorem 7.2.

To tie up with remarks made after Theorem 3.9, we extract the next result from the proof of Theorem 7.2.

PROPOSITION 7.4. *Let M be a finite noncyclic A_4-submodule of B_2. Then $|H^2(A_4, M)| = 1$ except when M is one of the following, in which cases $|H^2(A_4, M)| = 2$:*

$$F(i,j,j,j,0,0), \quad F(i,j,j,j,0,0,-1,-1), \quad F(i,j,j,j,0,0,0,-1).$$

Proof. When compiling $\mathscr{A}$, we found that the number of $D(4)A_4$-conjugacy classes of A_4-extensions of M is either 6 (for those M such that $|H^2(A_4, M)|$ is claimed to be 2) or 3 (for all other M). Then Theorem 3.9(ii) yields the result. ∎

8. THE SUBLIST FOR $T = S_4$

THEOREM 8.1. *Define the list $\mathscr{S}$ of finite irreducible subgroups of $M(4)$ to consist of all semidirect products $\hat{G}N$, where $\hat{G}$ is one of the following (ε, η range over $\{0,1\}$ and ν ranges over $\{-1,1\}$), and N is any finite S_4-submodule of $B_{2'}$, unless stated otherwise.*

For $i, j \geq 1$,

$$\left\langle a, b, d, ex_{i+1,2}^{\varepsilon}(y_{j+1,2}u_{j+1,2}v_{j+1,2})^{\eta}, F(i,j,j,j,0,0)\right\rangle,$$

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}, F(i,j,j,j,0,0,-1,-1)\rangle,$$

$$\Big\langle ax_{i+1,2}, bu_{j+1,2}, du_{j+2,2}v_{j+2,2}^{-1}, ex_{i+2,2}^{\nu}y_{j+2,2}u_{j+1,2}, \\ F(i,j,j,j,0,0,-1,-1)\Big\rangle,$$

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}, F(i,j,j,j,0,0,0,-1)\rangle,$$

$$\langle a, bu_{j+1,2}, du_{j+2,2}v_{j+2,2}^{-1}, ex_{i+1,2}^{\varepsilon}y_{j+2,2}u_{j+1,2}, F(i,j,j,j,0,0,0,-1)\rangle,$$

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}y_{j+1,2}^{\eta}, F(i,j,j,j,0,1,0)\rangle.$$

For $i, j \geq 1$ or $i = j = 0$,

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}, F(i,j,j,j,1,1,1)\rangle,$$

$$\langle a, b, d, ex_{i+2,2}^{\nu}y_{j+2,2}u_{j+2,2}v_{j+2,2}, F(i,j,j,j,1,1,1)\rangle,$$

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}, F(i,j,j,j,1,1,2,1)\rangle,$$

$$\langle a, b, d, ex_{i+1,2}^{\varepsilon}, F(i,j,j,j,1,1,2,-1)\rangle.$$

For $i \geq -1$ and N such that for some prime $p \geq 3$, $N_p = F(i', j', j', j'; p)$ with $j' \geq 0$,

$$\langle a, b, d, ex^{\varepsilon}_{i+1,2}, M(i, -1, -1, -1)\rangle .$$

For $i \geq 0$,

$$\langle ax_{1,2}, bu_{1,2}, du_{2,2}v^{-1}_{2,2}, ex^{\varepsilon}_{i+1,2}x_{2,2}y_{2,2}u_{1,2}, M(i, -1, -1, -1)\rangle .$$

If G is a finite irreducible subgroup of $M(4)$ with projection group S_4, then G is $M(4)$-conjugate to one and only one group in $\mathscr{S}$.

Proof. Denote the diagonal subgroup of G by M. By Theorem 3.11(ii), G is $M(4)$-conjugate to $\langle exyuv, K, M_3\rangle \ltimes M_{\{2,3\}'}$ for some $x \in X$, $y \in Y$, $u \in U$, $v \in V$, where K has projection group A_4 and diagonal subgroup M_2. For any finite faithful S_4-submodule L of $B_{\{2,3\}'}$, KM_3L is irreducible, so that KM_3L is $M(4)$-conjugate to a group in $\mathscr{A}$. Therefore $KM_{2'}$ is $M(4)$-conjugate to a group in the list obtained from $\mathscr{A}$ by relaxing the restrictions on M_3 and N in Theorem 7.2 that hold for cyclic M_2. Conditions derived from the relations involving e in the chosen presentation of S_4 may be written down; these dictate the choices for $xyuv$ and for K from the list in Theorem 7.2. (For example, $xyuv$ is in B_2, and the condition derived from the relation $d^e = d^2$ rules out groups not in B_2A_4 as possible K.) Conversely, if these conditions are satisfied for given K, then $\langle exyuv, K\rangle$ is an S_4-extension of M_2 in $M(4)$. We do not give any more of this sort of detail in the proof that $\langle exyuv, K\rangle$ is $M(4)$-conjugate to one of the subgroups $\hat{G}$ of B_2S_4 as described. In any event, the matter of irreducibility has to be settled. When M_2 is cyclic, G is irreducible if and only if $N = M_{2'}$ is faithful, or $\hat{G}$ is irreducible. If

$$\hat{G} = \langle ax_{1,2}, bu_{1,2}, du_{2,2}v^{-1}_{2,2}, ex^{\varepsilon}_{i+1,2}x_{2,2}y_{2,2}u_{1,2}, M(i, -1, -1, -1)\rangle ,$$

then $\mathsf{C}_{GL(4)}(\hat{G}) = X$, so $\hat{G}$ is irreducible and N is unrestricted for these $\hat{G}$. On the other hand,

$$\hat{G} = \langle a, b, d, ex^{\varepsilon}_{i+1,2}, M(i, -1, -1, -1)\rangle$$

is reducible, and N is faithful by the stated restriction. When M_2 is noncyclic, M is a faithful S_4-module and so G is irreducible by Theorem 4.2.

We have demonstrated that G is conjugate to a group in $\mathscr{S}$. Next we consider whether groups in $\mathscr{S}$ are $M(4)$-conjugate. Suppose $\hat{G}N \in \mathscr{S}$ and $(\hat{G}N)^m = \hat{G}^m N \in \mathscr{S}$ for some $m \in M(4)$. If M_2 is cyclic then $\hat{G} = \hat{G}^m$ by standard cohomology-based arguments. So from now on, M_2 is noncyclic.

We see that $\hat{G} \cap B_2 D$ and $\hat{G}^m \cap B_2 D$, as Sylow 2-subgroups of groups conjugate in $M(4)$, are themselves $M(4)$-conjugate. It is easy enough to determine these Sylow 2-subgroups from the generating sets given in $\mathscr{S}$. However, to recognise the groups in the list of [5, Theorem 5.7], one has to translate between two different labelling schemes for D-submodules of B. To illustrate: if

$$\hat{G} = \left\langle a, b, d, ex^{\varepsilon}_{i+1,2}(y_{j+1,2}u_{j+1,2}v_{j+1,2})^{\eta}, F(i,j,j,j,0,0) \right\rangle$$

for some fixed values of η and ε, then we have

$$\hat{G} \cap B_2 D = \left\langle a, cx^{\varepsilon}_{i+1,2}(y_{j+1,2}u_{j+1,2}v_{j+1,2})^{\eta}, C(i,j,j,0,0) \right\rangle.$$

Further,

$$\hat{G} \cap B_2 D \sim_{(u_{j+2,2}y_{j+1,2})^{\eta}} \left\langle au^{\eta}_{j+1,2}, cx^{\varepsilon}_{i+1,2}, C(i,j,j,0,0) \right\rangle.$$

Since $\hat{G}^m$ and $\hat{G}$ have the same diagonal subgroup, $\hat{G}^m \cap B_2 D$ must also be B_2-conjugate to one of the $\langle au^{\eta}_{j+1,2}, cx^{\varepsilon}_{i+1,2}, C(i,j,j,0,0)\rangle$. But by [5, Theorem 5.7], for fixed i and j the pair of values (η, ε) determines a unique $M(4)$-conjugacy class of such groups. Thus $\hat{G} \cap B_2 D = \hat{G}^m \cap B_2 D$. Indeed, with the aid of [5, Theorem 5.7], it may be seen in the same fashion that $\hat{G} \cap B_2 D = \hat{G}^m \cap B_2 D$ for all $\hat{G}$ appearing in $\mathscr{S}$ such that $\hat{G} \cap B_2$ is noncyclic (we suppress the calculations). It is clear from inspection that a group in $\mathscr{S}$ is determined by its Sylow 2-subgroup in $B_2 D$. Hence $\hat{G} = \hat{G}^m$, and distinct groups in $\mathscr{S}$ are not $M(4)$-conjugate. This completes our proof of the theorem. ∎

We state next an analogue of Proposition 7.4.

PROPOSITION 8.2. *Let M be a finite noncyclic S_4-submodule of B_2. Then $|H^2(S_4, M)| = 4$ except when $M = F(i,j,j,j,1,1,2,\pm 1)$, in which cases $|H^2(S_4, M)| = 2$.*

The import of the final result of this section is that we have already solved the full conjugacy problem for groups in $\mathscr{S}$.

THEOREM 8.3. *$\mathscr{S}$ is a complete and irredundant list of $GL(4)$-conjugacy class representatives of the finite irreducible subgroups of $M(4)$ with projection group S_4.*

Proof. Suppose $G, H \in \mathscr{S}$ and $G^m = H$ for some $m \in GL(4)$. Set $M = B \cap G$. Denote by $\tilde{G}$ the subgroup of G containing M such that $\pi\tilde{G} = A_4$.

Let $K \neq \tilde{G}$ be a subgroup of G with index 2 and an abelian subgroup N normal in G such that $K/N \cong A_4$. Then $M \not\leq K$ and $|M : M_1| = 2$, where $M_1 = M \cap K$. Now NM_1/N is an abelian normal subgroup of K/N; it cannot be trivial, because if $M_1 \leq N$ then NM/M is a normal subgroup of $G/M \cong S_4$ of order 2. Thus $NM_1/N \cong V_4$ and $N \cap M$ has index 8 in N. But then πN is a normal subgroup of S_4 of order 8. Hence $K = \tilde{G}$: this is the unique subgroup of its kind in G.

Since $M_{2'}^m = H \cap B_{2'} = M_{2'}$, if $M_{2'}$ is a faithful S_4-module then m is monomial by Proposition 5.4(ii), and we are done by Theorem 8.1. So we restrict attention to the cases that M_2 is a faithful S_4-module, or M is scalar and $G \cap B_2 S_4$ is irreducible.

Suppose M_2 is faithful. We assume $M_2^m \not\leq B$, since otherwise m is monomial and there is nothing to prove. Thus $\pi M_2^m = \langle a, b \rangle$ and M_2 has a scalar subgroup Q of index 4. Consequently $M_2 = \langle x_{1,2} y_{1,2}, x_{1,2} u_{1,2} \rangle \times Q$ is either $F(0,0,0,0,1,1,1)$ of order 8, or $F(i,1,1,1,0,0)$, some $i \geq 1$, of order 2^{i+3}. Since $|M_2| = |B_2 \cap H|$, we must have $M_2 = B_2 \cap H$. Moreover, $M = B \cap H$. By Theorem 8.1 and what was proved in the second paragraph, m normalises $\tilde{G} = \langle a, b, d, M \rangle$. Now we observe that $\langle x_{1,2} y_{1,2}, x_{1,2} u_{1,2} \rangle^m$, as a complement of B_2 in $B_2 V_4$, is B_2-conjugate to V_4. Since each automorphism of V_4 is realised as a linear isomorphism induced by an element of $S_3 = \langle d, e \rangle$, it follows that $(x_{1,2} y_{1,2})^{mz} = a$ and $(x_{1,2} u_{1,2})^{mz} = b$ for some $z \in B_2 S_3$. By simple matrix multiplication, this gives $mz \in D(4)n$, where n is the Hadamard matrix

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

But one readily checks that $(e y_{2,2} u_{2,2} v_{2,2})^{z'n} \notin M(4)$ for any $z' \in D(4)$. Therefore, if $\tilde{G} \cap B_2 = F(i,1,1,1,0,0)$ then only $\langle e, \tilde{G} \rangle$ and $\langle e x_{i+1,2}, \tilde{G} \rangle$ of the (different) groups in the list of Theorem 8.1 could possibly be conjugate; and if $\tilde{G} \cap B_2 = F(0,0,0,0,1,1,1)$ then only $\langle e, \tilde{G} \rangle$ and $\langle e x_{1,2}, \tilde{G} \rangle$ could be conjugate. However, for fixed $\tilde{G}$, the second listed group in each situation does not split over $\tilde{G}$.

Suppose M is scalar and $G \cap B_2 S_4$ is irreducible. By Theorem 8.1, the choices for G and H are

$$\langle a x_{1,2}, b u_{1,2}, d u_{2,2} v_{2,2}^{-1}, e x_{2,2} y_{2,2} u_{1,2}, M \rangle,$$

$$\langle a x_{1,2}, b u_{1,2}, d u_{2,2} v_{2,2}^{-1}, e x_{i+1,2} x_{2,2} y_{2,2} u_{1,2}, M \rangle,$$

where $M_2 = M(i, -1, -1, -1)$ for some $i \geq 0$. (Note that each of these groups normalises the other.) Now m induces an isomorphism $\overline{m}$ of G/M onto H/M that leaves $\tilde{G}/M = \langle ax_{1,2}, bu_{1,2}, du_{2,2}v_{2,2}^{-1}, M\rangle/M \cong A_4$ invariant. Hence, there exists $g \in G$ such that $\overline{m}$ on $\tilde{G}/M$ has the same action as the inner automorphism $\bar{g}$ of G/M induced by gM. Since it is the identity on A_4, $\overline{m}\bar{g}^{-1}$ is the identity on S_4; more precisely, $\pi(h^{mg^{-1}}) = \pi h$ for all $h \in G$. Given the choices for G and H, and the fact that there is no invertible matrix mg^{-1} such that $(ey_{2,2}u_{1,2})^{mg^{-1}} \equiv ex_{i+1,2}y_{2,2}u_{1,2} \bmod M$, the only feasible option is $G = H$. ∎

9. CONSOLIDATION

We have produced five separate lists $\mathscr{F}, \mathscr{C}, \mathscr{D}, \mathscr{A}, \mathscr{S}$, of finite irreducible subgroups of $M(4)$. We have seen that a given finite irreducible subgroup of $M(4)$ is conjugate to at least one group in the union of these lists, and also that distinct groups in $\mathscr{F} \cup \mathscr{C} \cup \mathscr{D}$ are not conjugate: this is just a reiteration of the substance of Theorems 6.1.4, 6.2.3, 6.3.3, 7.2, and 8.1. By Theorems 7.2 and 8.3 there is no conjugacy between distinct groups in $\mathscr{A}$ nor between distinct groups in $\mathscr{S}$. We show next that no group in $\mathscr{A}$ is conjugate to a group in $\mathscr{S}$, nor is any group in $\mathscr{A} \cup \mathscr{S}$ conjugate to a group in $\mathscr{F} \cup \mathscr{C} \cup \mathscr{D}$.

PROPOSITION 9.1. *Let G, H, K be finite subgroups of BS_4, where $\pi G \leq D$, $\pi H = A_4$, and $\pi K = S_4$. Then G, H, K are pairwise non-isomorphic.*

Proof. Let R stand for G or H, and S stand for H or K, where $R \neq S$. Suppose $R \cong S$. Then S has an abelian normal subgroup N such that $S/N \cong R/(B \cap R)$. Denote the diagonal subgroup of S by M. Since MN/M is a nontrivial normal abelian subgroup of S/M, its index is 3 or 6. If $R = G$ then this contradicts the fact that S/MN, as a quotient of S/N, has 2-power order. If $R = H$ then $S = K$ and $|S : MN| = 6$. This implies MN/N is a normal subgroup of $S/N \cong A_4$ of order 2, which is absurd. ∎

Linking Proposition 9.1 with the above-mentioned theorems, we obtain the following solution of the main problem of this paper.

THEOREM 9.2. *A finite irreducible monomial subgroup of $GL(4)$ is conjugate to one and only one group in $\mathscr{F} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{A} \cup \mathscr{S}$.*

Remark 9.3. Given the amount of data involved and the nature of supplementary results (such as those discussed before Theorem 2.3.3 and Lemma 6.1.2), we have decided to make the list of Theorem 9.2 available in electronic form, as a library in the computer algebra system GAP. At the time of writing, the GAP library is under construction, as joint work of

B. Höfling and the author. We refer the interested reader to `http://www.minet.uni-jena.de/~hoefling/irreducible.html` for more details.

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